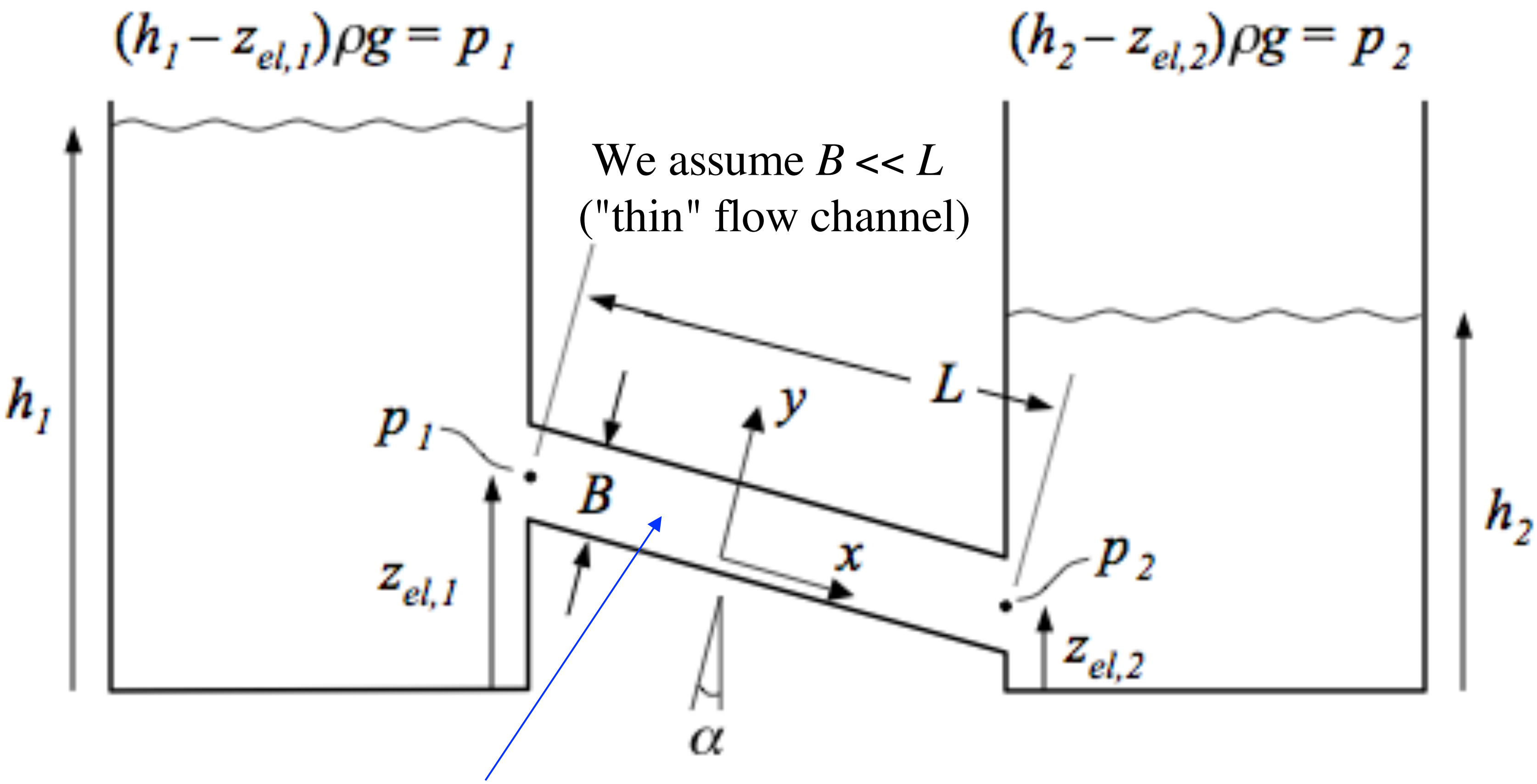
Engineering Sciences 220, Fall Term 2014
Fluid Dynamics

Dynamics of Viscous Incompressible Fluids

Visuals 02

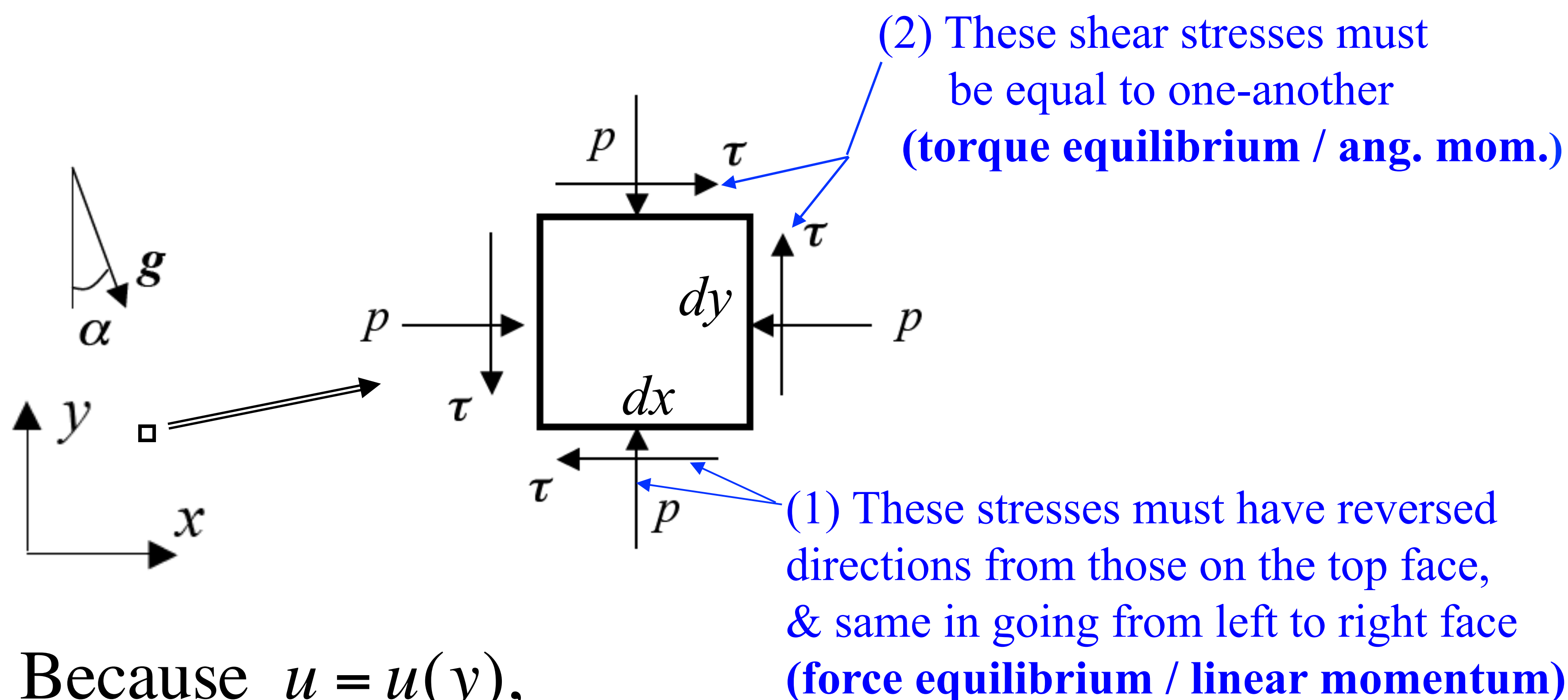
J. R. Rice, Fall term 2014

Flow in a Very Narrow Slit, Driven by a Gradient in Head



We assume flow is ***laminar***, with velocity u in the x-direction, the same at each cross section, so that $u = u(y)$. Recall that ***shear rate*** $= du(y)/dy$

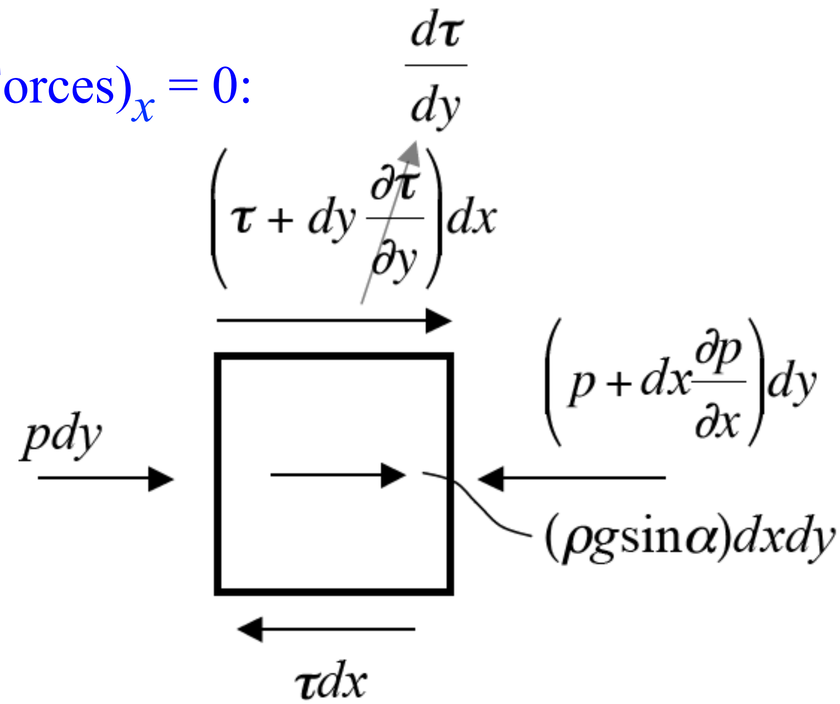
Stresses and equilibrium



Because $u = u(y)$,

$$\tau = \mu \frac{du(y)}{dy} = \tau(y)$$

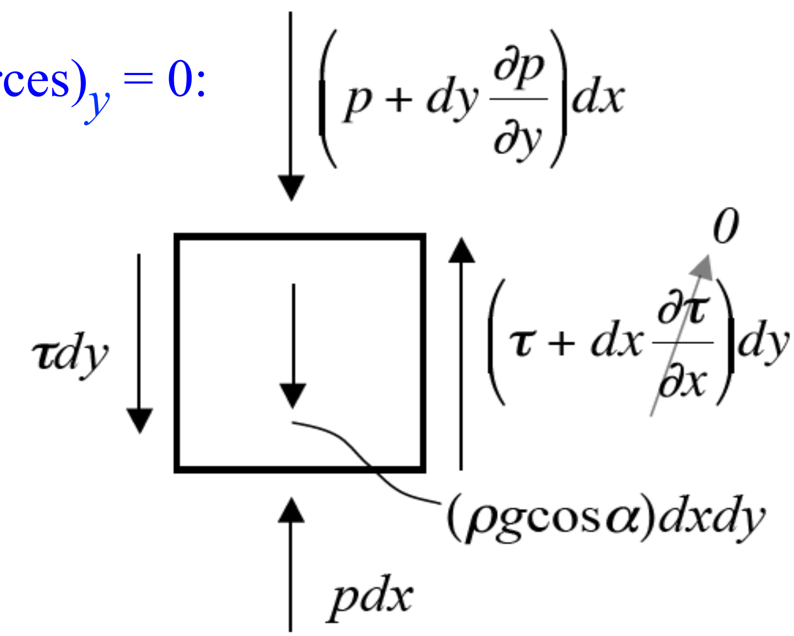
Σ (Forces)$_x$ = 0:



$$pdy - \left(p + dx\frac{\partial p}{\partial x}\right)dy$$
$$-\tau dx + \left(\tau + dy\frac{\partial \tau}{\partial y}\right)dx$$
$$+(\rho g \sin\alpha)dxdy = 0$$

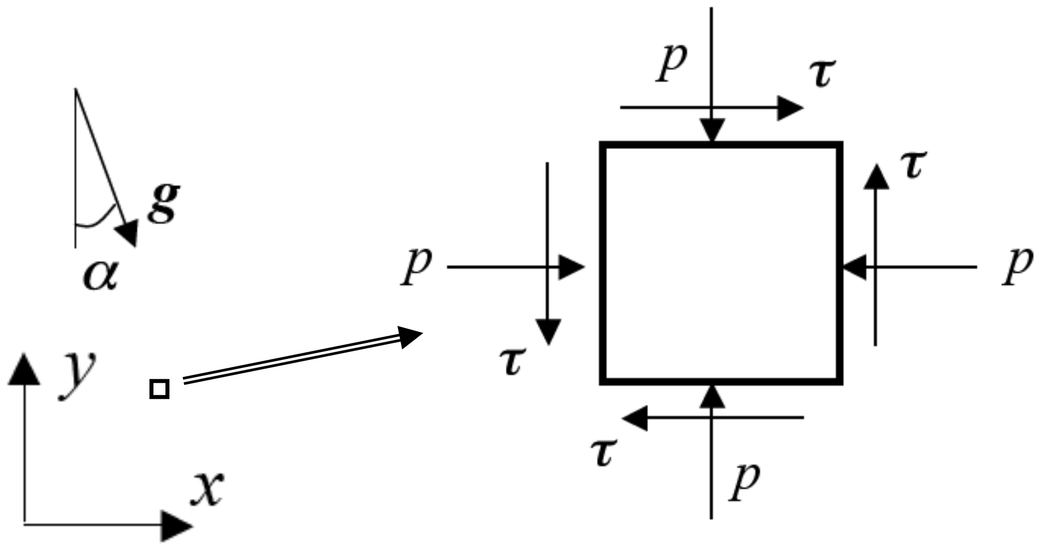
$$\Rightarrow -\frac{\partial p}{\partial x} + \frac{d\tau(y)}{dy} + \rho g \sin\alpha = 0$$

Σ (Forces)$_y$ = 0:



$$-\tau dy + (\tau + dx\frac{\partial \tau}{\partial x})dy$$
$$+ pdx - (p + dy\frac{\partial p}{\partial y})dx$$
$$-(\rho g \cos\alpha)dxdy = 0$$

$$\Rightarrow -\frac{\partial p}{\partial y} - \rho g \cos\alpha = 0$$



2D Equilibrium equations, in general, are

$$-\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial y}+\rho g\sin\alpha=0\ , \qquad -\frac{\partial p}{\partial y}+\frac{\partial \tau}{\partial x}-\rho g\cos\alpha=0$$

For our problem, $u=u(y)$ *and* $\tau=\mu\dfrac{du(y)}{dy}=\tau(y)$, *so they reduce to*

$$-\frac{\partial p}{\partial x}+\frac{d\tau(y)}{dy}+\rho g\sin\alpha=0\ , \qquad -\frac{\partial p}{\partial y}-\rho g\cos\alpha=0$$

$$(1)\quad -\frac{\partial p}{\partial x}+\frac{d\tau(y)}{dy}+\rho g\sin\alpha=0\ ,\qquad (2)\quad -\frac{\partial p}{\partial y}-\rho g\cos\alpha=0$$

Taking $\frac{\partial}{\partial x}$ of Eq.(2) $\Rightarrow$ $\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)=0$

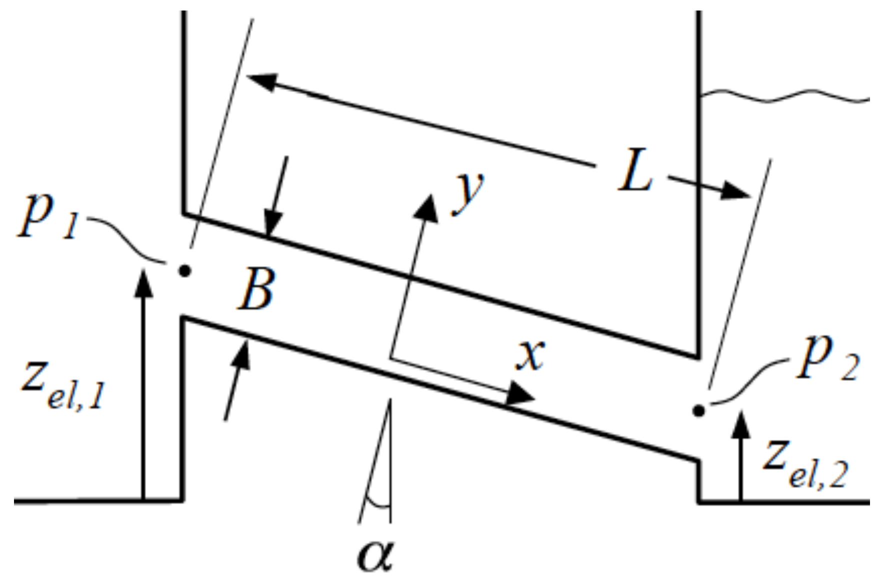
$\Rightarrow$ $\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)=0$ $\left[\text{Recall that } \frac{\partial^2 p}{\partial x\partial y}=\frac{\partial^2 p}{\partial y\partial x}\right]$

But taking $\frac{\partial}{\partial x}$ of Eq.(1) $\Rightarrow$ $\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right)=0$.

$$\therefore\ \frac{\partial p}{\partial x}=const.\equiv\frac{p_2-p_1}{L}$$

Also, from (2),

$$\frac{\partial p}{\partial y}=const.\equiv-\rho g\cos\alpha$$



Now we use eq. (1), $-\dfrac{\partial p}{\partial x} + \dfrac{d\tau(y)}{dy} + \rho g \sin\alpha = 0$, and set

$\tau(y) = \mu \dfrac{du(y)}{dy}$ to give eq. (3), $\mu \dfrac{d^2u(y)}{dy^2} = \dfrac{\partial p}{\partial x} - \rho g \sin\alpha$.

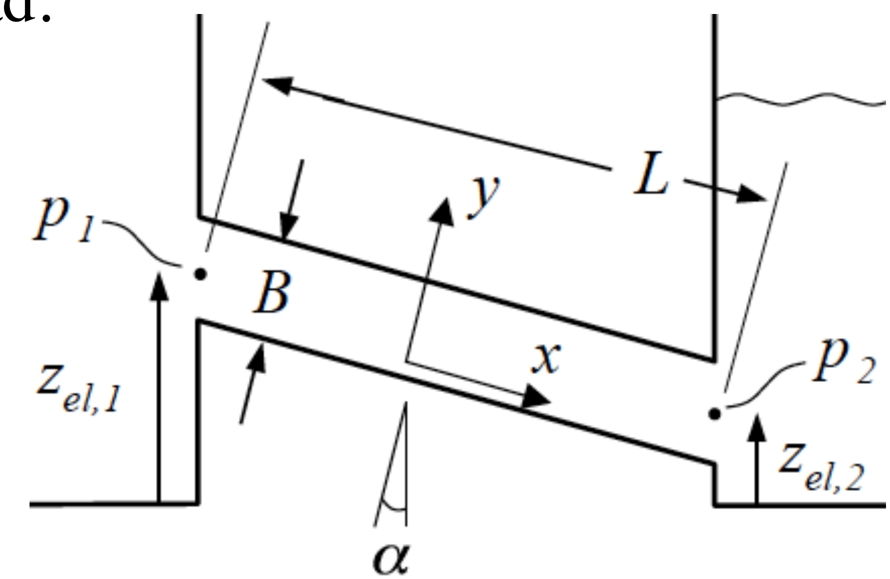
Then recall that $\dfrac{\partial p}{\partial x} = \dfrac{p_2 - p_1}{L}$, and note that $\sin\alpha = -\dfrac{z_{el,2} - z_{el,1}}{L}$,

so the R.H.S. of eq. (3) is $\dfrac{\partial p}{\partial x} - \rho g \sin\alpha = \rho g \dfrac{h_2 - h_1}{L} = \rho g \dfrac{\partial h}{\partial x}$,

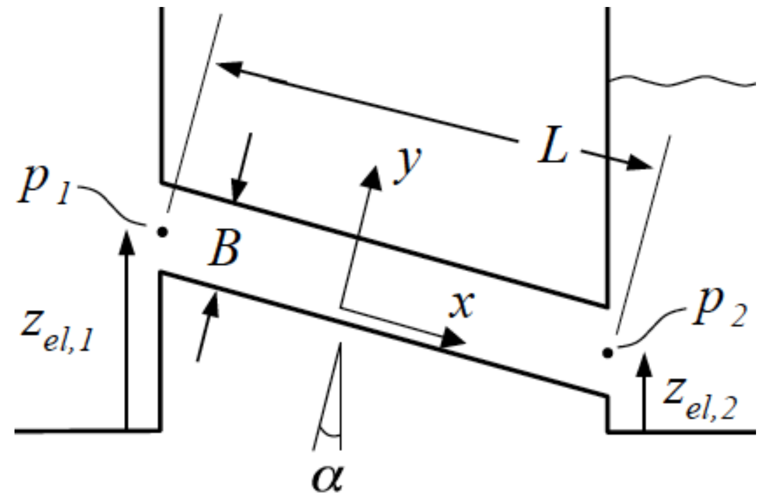
where $h = \dfrac{p}{\rho g} + z_{el}$ = hydraulic head.

Thus, $\mu \dfrac{d^2u(y)}{dy^2} = \rho g \dfrac{\partial h}{\partial x}$.

(It is not gravity or pressure drop alone which drives flow, but rather their combination as the head drop.)



$$\frac{d^2u(y)}{dy^2} = \frac{\rho g}{\mu}\frac{\partial h}{\partial x}$$

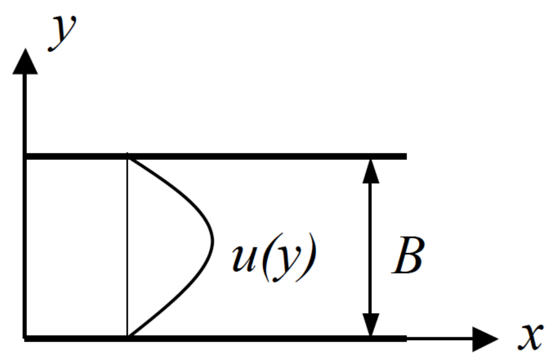


$$\therefore\ u(y) = \frac{\rho g}{2\mu}\frac{\partial h}{\partial x}[(y - B/2)^2 + C_1(y - B/2) + C_2]$$

$$\left.\begin{array}{l}\text{At } y = 0,\ u = 0 \ \Rightarrow\ (B/2)^2 - C_1(B/2) + C_2 = 0 \\ \text{At } y = B,\ u = 0 \ \Rightarrow\ (B/2)^2 + C_1(B/2) + C_2 = 0\end{array}\right\| \Rightarrow\ C_1 = 0,\ C_2 = -(B/2)^2$$

$$\therefore\ u(y) = -\frac{\rho g B^2}{8\mu}\frac{\partial h}{\partial x}\left(1 - \left(\frac{y - B/2}{B/2}\right)^2\right)$$

$$u(y) = -\frac{\rho g B^2}{8\mu}\frac{\partial h}{\partial x}\left(1-\left(\frac{y-B/2}{B/2}\right)^2\right) = -\frac{gB^2}{8\nu}\frac{\partial h}{\partial x}\left(1-\left(\frac{y-B/2}{B/2}\right)^2\right) \quad \left(\nu \equiv \frac{\mu}{\rho} \left[\frac{\mathrm{L}^2}{\mathrm{T}}\right]\right)$$



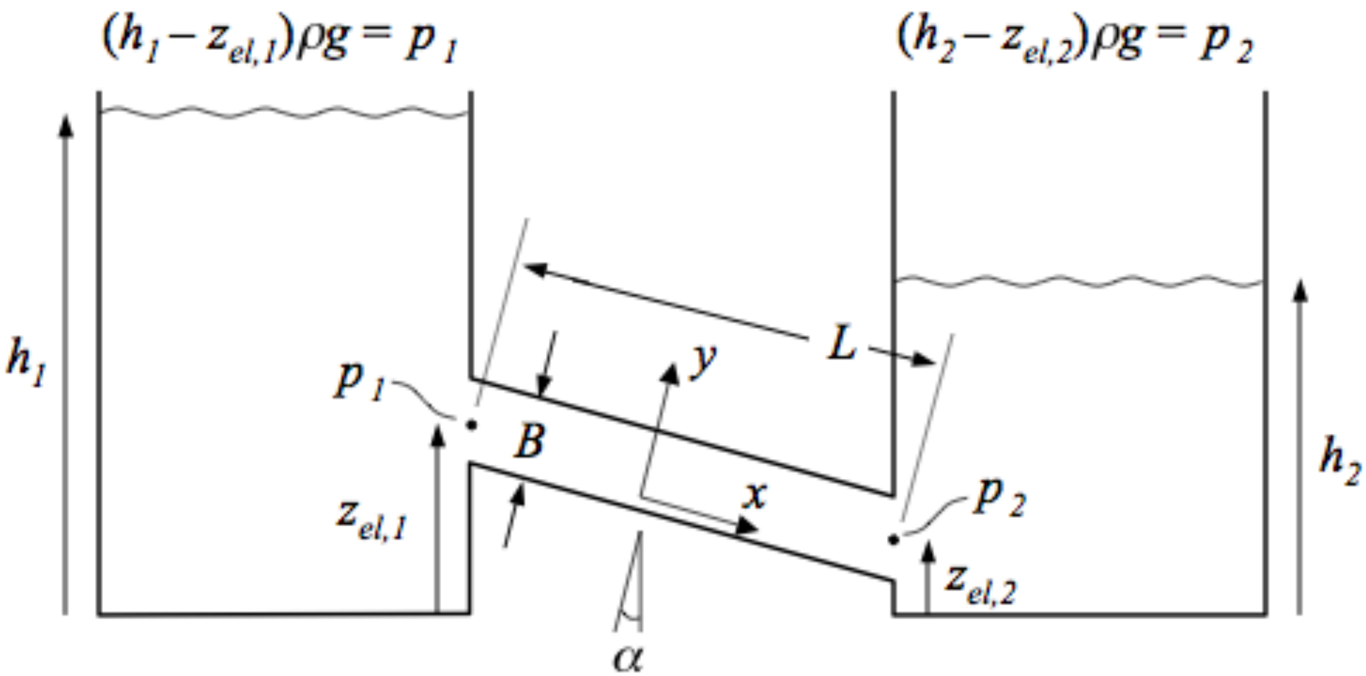
Average velocity U:

$$U = \frac{1}{B}\int_0^B u(y)dy = -\frac{gB^2}{12\nu}\frac{\partial h}{\partial x}$$

Volumetric flow rate Q per unit distance in z direction:

$$Q = UB = -\frac{gB^3}{12\nu}\frac{\partial h}{\partial x}$$

Solution in terms of gradient in head



$$-\frac{dh}{dx} = \text{constant} = \frac{h_1 - h_2}{L} \quad \text{within slit}$$

$$u(y) = -\frac{\rho g B^2}{8\mu}\frac{dh}{dx}\left(1 - \left(\frac{y - B/2}{B/2}\right)^2\right)$$

$$U = -\frac{\rho g B^2}{12\mu}\frac{dh}{dx} \quad \text{and} \quad Q \equiv BU = -\frac{\rho g B^3}{12\mu}\frac{dh}{dx}$$

Flow Through (Inclined) Circular Tubes

- Similar analysis to that for flow in a slit between parallel walls.
- Seek solution with $u = u(r)$;
- Note that $u = u(r)$ implies that $\tau = \tau(r) = \mu\, du(r)/dr$.
- Applying equations of equilibrium in directions *perpendicular* to flow direction shows that $h \equiv (p/\rho g) + z_{el} = h(x)$, i.e., h is constant in each cross section (show this, i.e., that equilibrium and $\tau = \tau(r)$ implies $\partial h/\partial y = \partial h/\partial z = 0$).
- Result h is constant in each cross section implies (why?) that $\partial p/\partial x$ is also constant -- although p itself varies within a cross section.
- Apply equation of equilibrium in x-direction to cylindrical fluid volume of arbitrary radius r (where $r \leq a$; a = tube radius) and length dx:

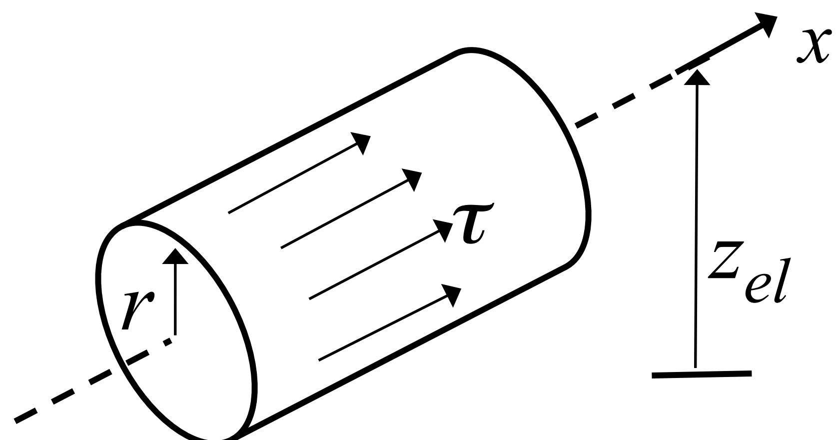
$$(2\pi r dx)\tau = (\pi r^2)(dx\frac{\partial p}{\partial x}) + (\pi r^2 dx)\rho g\frac{\partial z_{el}}{\partial x}$$

$$\Rightarrow \tau = \frac{\rho g r}{2}\frac{\partial}{\partial x}\left(\frac{p}{\rho g} + z_{el}\right) = \frac{\rho g r}{2}\frac{\partial h}{\partial x} = \frac{\rho g r}{2}\frac{dh(x)}{dx}$$

- $\tau = \tau(r) \;\Rightarrow\; \dfrac{dh}{dx}$ = const. (indep of x)
- $\tau = \mu\dfrac{du(r)}{dr} \;\Rightarrow\; \mu\dfrac{du(r)}{dr} = \dfrac{\rho g r}{2}\dfrac{dh}{dx}$

  Integrate, with $u = 0$ at $r = a$ to get

$$u = -\frac{\rho g}{4\mu}\frac{dh}{dx}\left(a^2 - r^2\right) \;\Rightarrow\; U = -\frac{\rho g a^2}{8\mu}\frac{dh}{dx}$$



Middleton and Southard, *Mechanics of Sediment Movement, 1984*

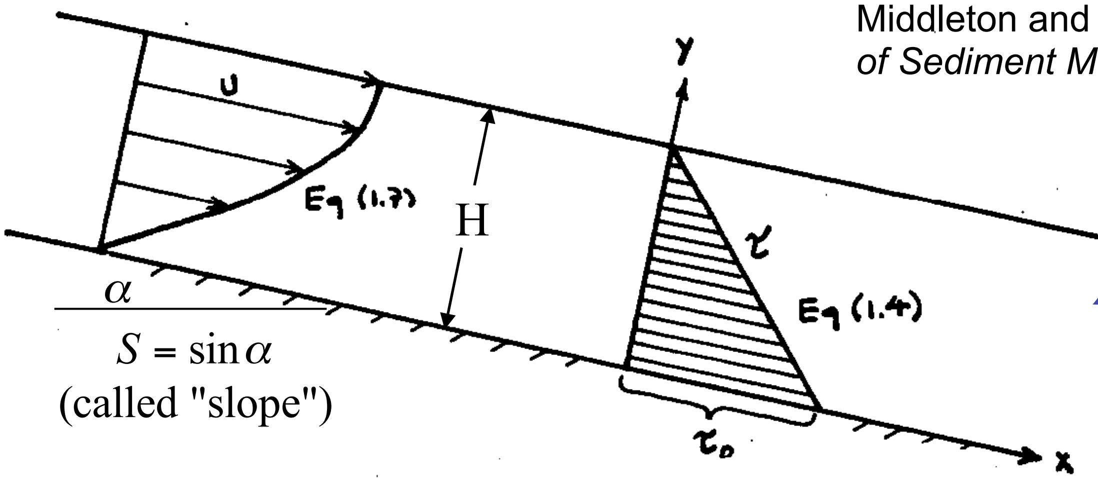


Figure 1.15 Distributions of (A) velocity and (B) shear stress in steady uniform laminar flow down an inclined plane.

Laminar flow:

$$U = gSH^2 / 3\nu$$

Solution same as for laminar ***flow in a slit***, with $H \to B/2$,

$$H - y \to B/2 - y$$

and $-\dfrac{dh}{dx} \to S$.

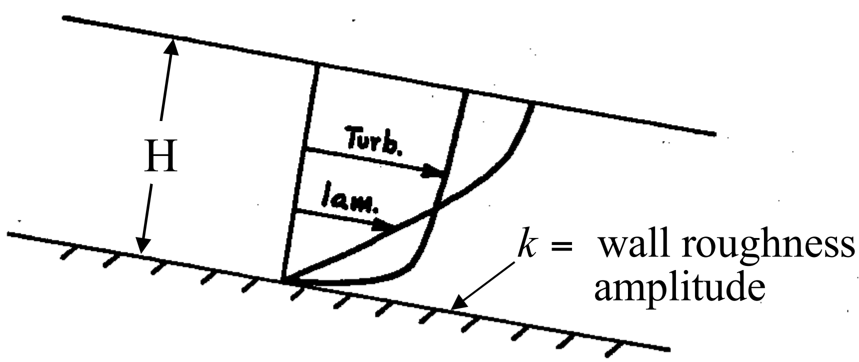


Figure 1.16 Comparison of velocity profiles in laminar and turbulent flow down an inclined plane. The profiles are drawn so that they represent about the same total discharge, or volume rate of flow, per unit width of the plane (that is, the areas under the two curves are about the same). There is a certain range of flow conditions for which the flow could be either laminar or turbulent depending on experimental conditions (although the slope needed for uniform flow would then be different in the two cases).

Middleton and Southard, *Mechanics of Sediment Movement, 1984*

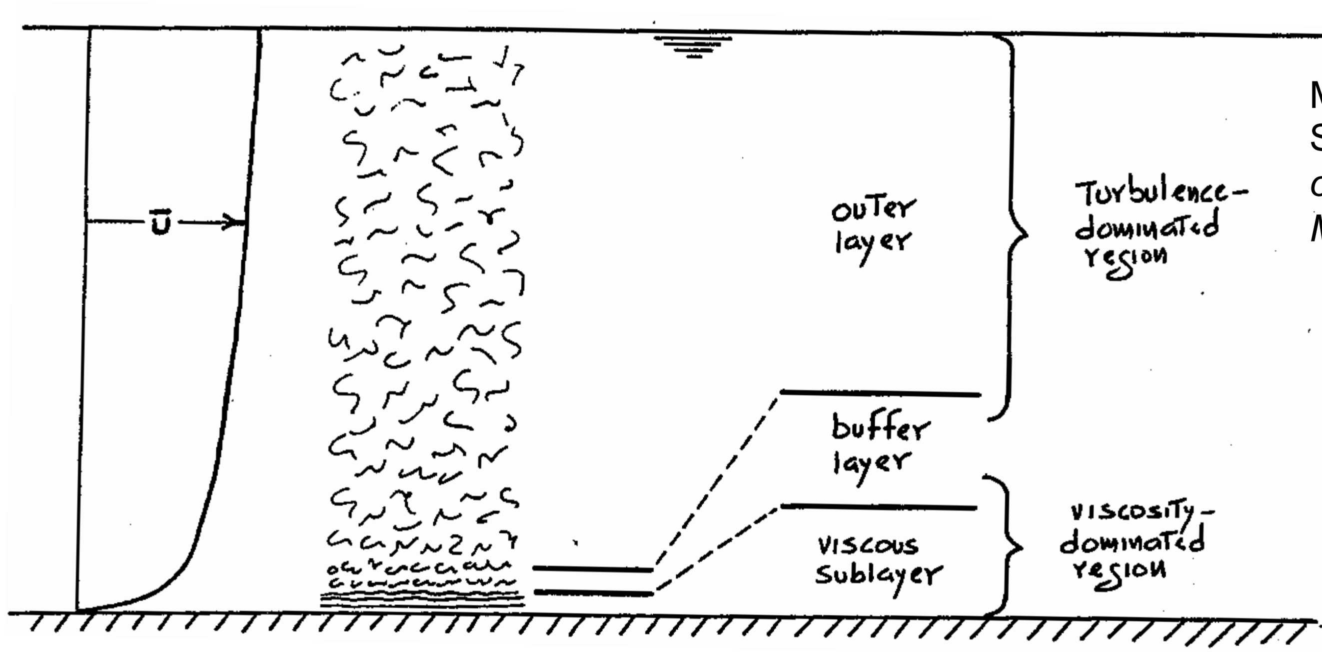


Figure 5.4 Zones of turbulence structure in steady uniform turbulent flow down a smooth plane. Eddy structure (schematic) is as seen by an observer moving with flow.

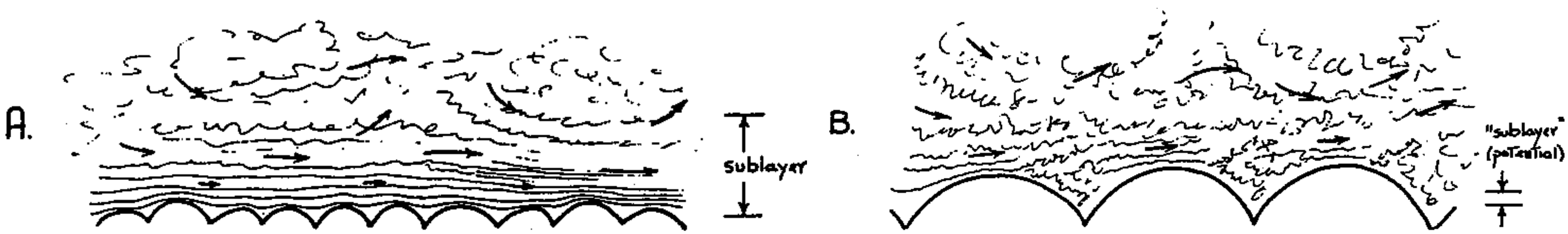
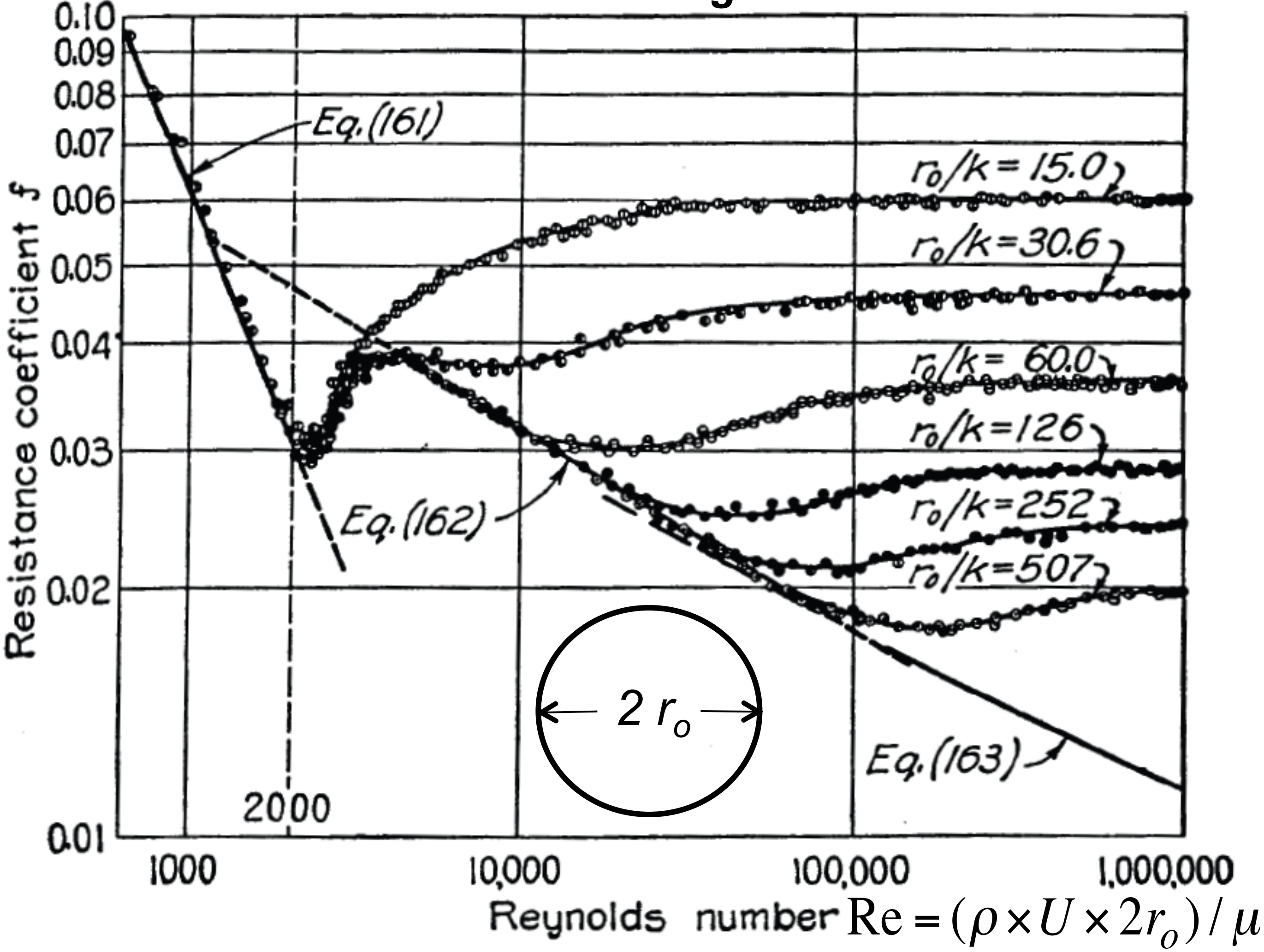


Figure 5.6 Relationship between viscous-sublayer thickness and height of roughness elements. (A) Roughness elements much smaller than sublayer thickness. (B) Roughness elements much larger than sublayer thickness.

Rouse, *Elementary Mechanics of Fluids*, 1946, 1978

Resistance to flow in rough-walled tube



Variation of the resistance coefficient with the Reynolds number for artificially roughened pipes. (r_o = pipe radius, k = roughness size)

Darcy - Wiesbach friction factor f :

$$f = \frac{\tau_{wall}}{\rho U^2 / 8}$$

In pipe flow:

$$f = \frac{Diam}{Length} \frac{\Delta p}{\rho U^2 / 2}$$

Re-interpretation based on hydraulic radius concept:

$$r_o / k$$

$$\rightarrow 2H / k$$

$\uparrow$

depth of very wide channel

Eq.(161) corresponds to Poiseuille solution, laminar flow in a circular tube, Homework Prob. 7(a)

Middleton and Southard, *Mechanics of Sediment Movement, 1984*

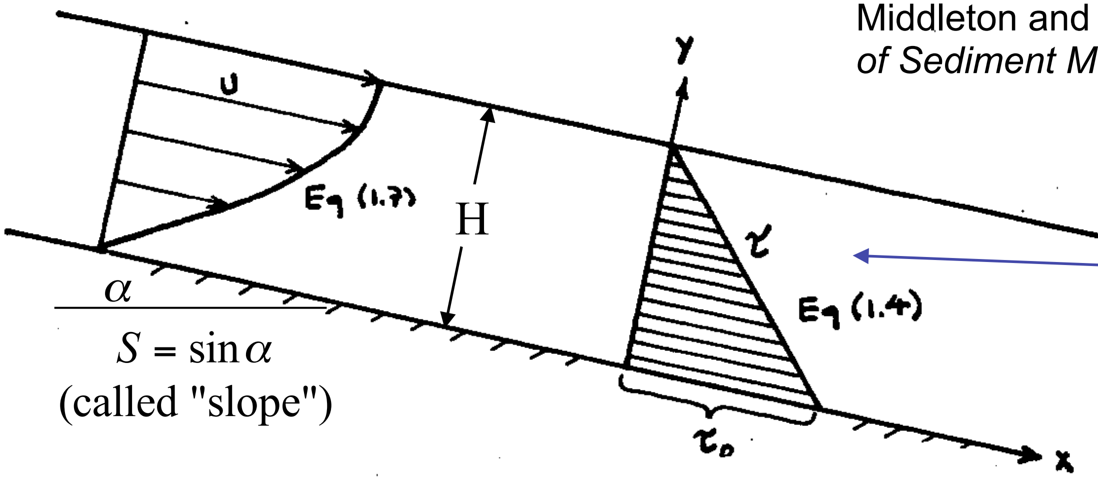


Figure 1.15 Distributions of (A) velocity and (B) shear stress in steady uniform laminar flow down an inclined plane.

Laminar flow:

$$U = gSH^2 / 3\nu$$

Turbulent flow (but with $Fr \equiv \dfrac{U}{(gH)^{1/2}}$ less than, and not very near to, 1):

Froude number

Reynolds number

$$\frac{U}{(gSH)^{1/2}} = \hat{f}\left(\mathrm{Re}, \frac{H}{k}\right) \quad \left(\mathrm{Re} \equiv \frac{UH}{\nu}\right);$$

$$\frac{U}{(gSH)^{1/2}} = \tilde{f}\left(\frac{(gS)^{1/2} H^{3/2}}{\nu}, \frac{H}{k}\right).$$

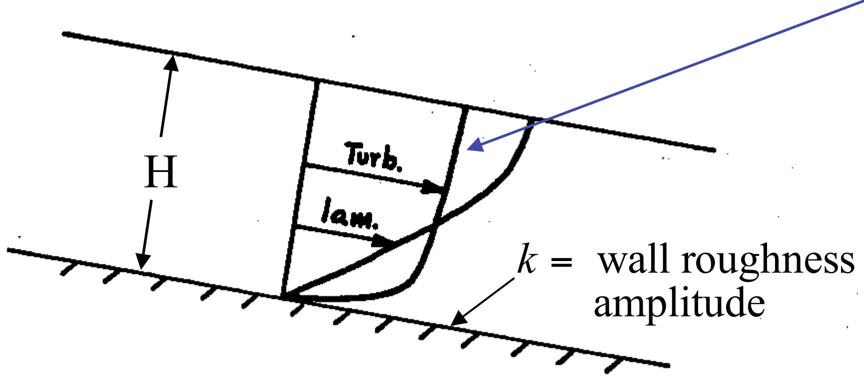
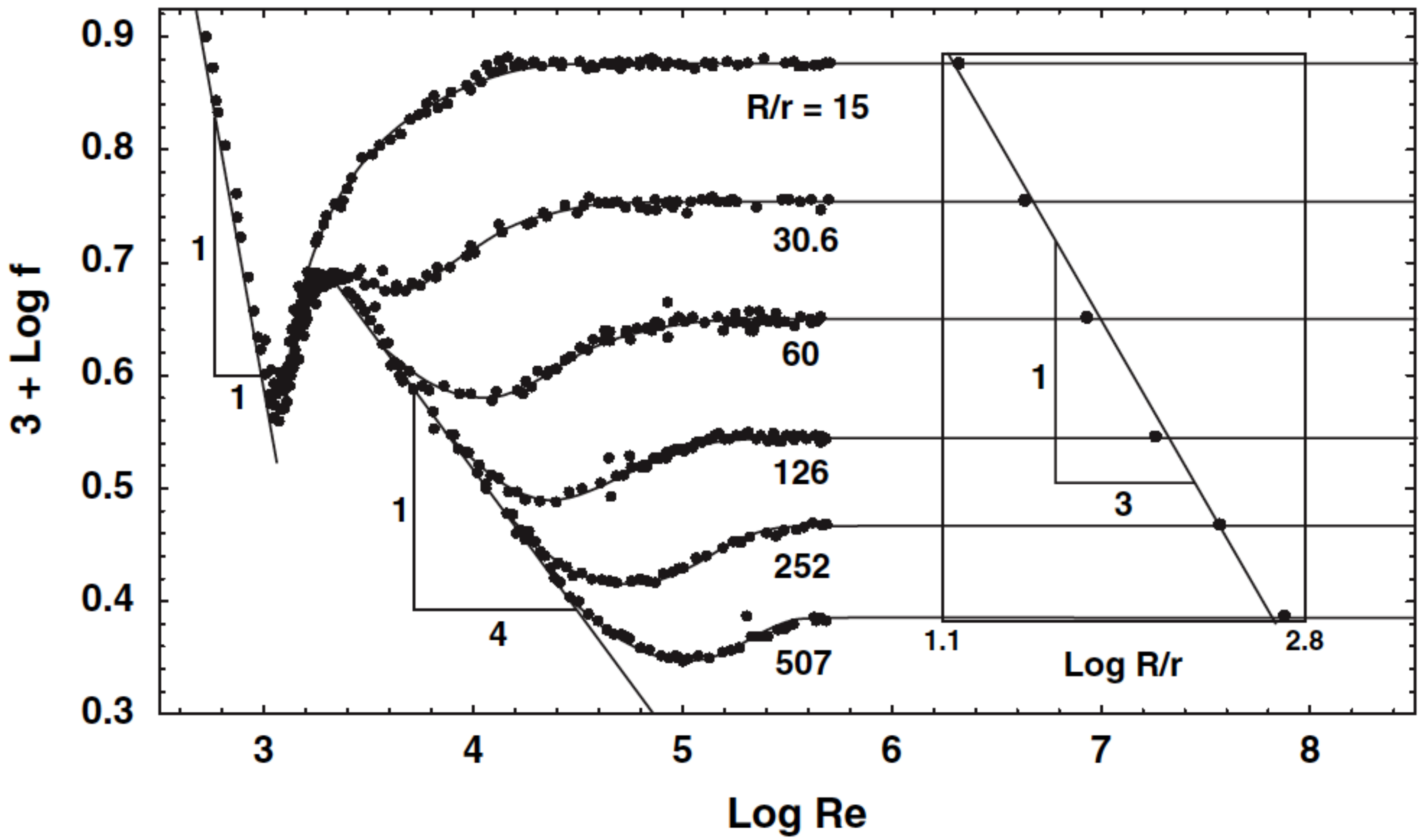


Figure 1.16 Comparison of velocity profiles in laminar and turbulent flow down an inclined plane. The profiles are drawn so that they represent about the same total discharge, or volume rate of flow, per unit width of the plane (that is, the areas under the two curves are about the same). There is a certain range of flow conditions for which the flow could be either laminar or turbulent depending on experimental conditions (although the slope needed for uniform flow would then be different in the two cases).

Gioia & Chakraborty [*PRL*, 2006] replot of Johann Nikuradse [1933] pipe-flow data



(r here is wall roughness k, R here (only!) is tube radius r_o)

$R/r \equiv r_o/k \rightarrow 2H/k$

Re = Reynolds number = $(\rho \times U \times 2r_o)/\mu = 2Ur_o/\nu$

FIG. 1. Nikuradse's data. Up to a Re of about 3000 the flow is streamlined (free from turbulence) and $f \sim 1/\text{Re}$. Note that for very rough pipes (small R/r) the curves do not form a belly at intermediate values of Re. Inset: verification of Strickler's empirical scaling for f at high Re, $f \sim (r/R)^{1/3}$.

Middleton and Southard, Mechanics of Sediment Movement, 1984

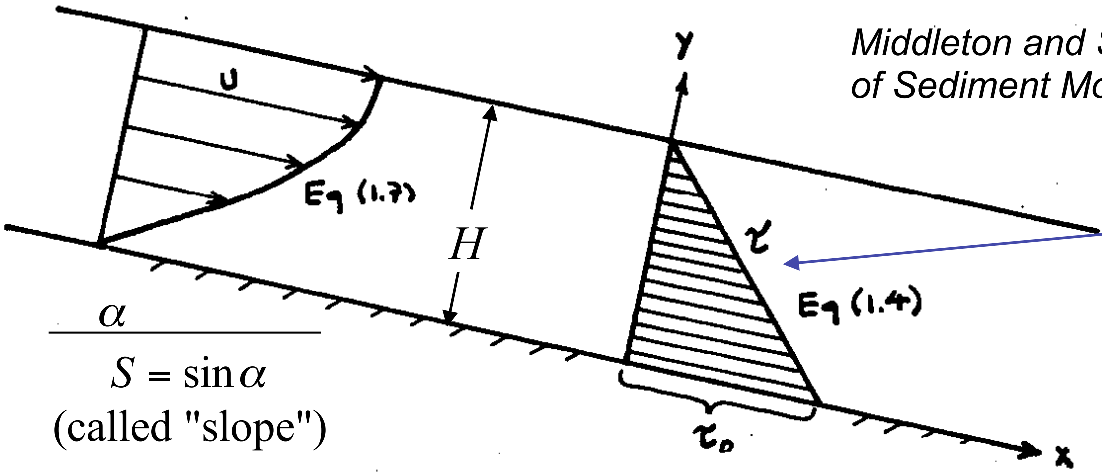


Figure 1.15 Distributions of (A) velocity and (B) shear stress in steady uniform laminar flow down an inclined plane.

Laminar flow:

$$U = gSH^2 / 3\nu$$

Turbulent flow (but with $\mathrm{Fr} \equiv \dfrac{U}{(gH)^{1/2}}$ (**Froude number**) less than, and not very near to, 1):

$$\frac{U}{(gSH)^{1/2}} = \hat{f}\left(\mathrm{Re}, \frac{H}{k}\right) \quad \left(\mathrm{Re} \equiv \frac{UH}{\nu}\right);$$

(**Reynolds number**)

$$\frac{U}{(gSH)^{1/2}} = \tilde{f}\left(\frac{(gS)^{1/2} H^{3/2}}{\nu}, \frac{H}{k}\right).$$

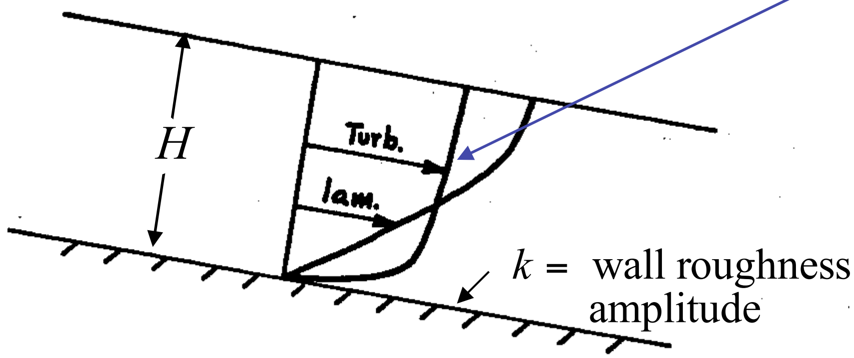


Figure 1.16 Comparison of velocity profiles in laminar and turbulent flow down an inclined plane. The profiles are drawn so that they represent about the same total discharge, or volume rate of flow, per unit width of the plane (that is, the areas under the two curves are about the same). There is a certain range of flow conditions for which the flow could be either laminar or turbulent depending on experimental conditions (although the slope needed for uniform flow would then be different in the two cases).

Empirically, at very large Re:

$$\frac{U}{(gSH)^{1/2}} \approx \tilde{f}\left(\frac{H}{k}\right) \approx const.\left(\frac{H}{k}\right)^{1/6}$$

[Manning,1891; Strickler,1923; + Prandtl vonKarman Nikuradse, 1930s]

$$U \approx H^{2/3} S^{1/2} / n_M \quad , \quad n_M \approx k^{1/6} / 8 g^{1/2}$$

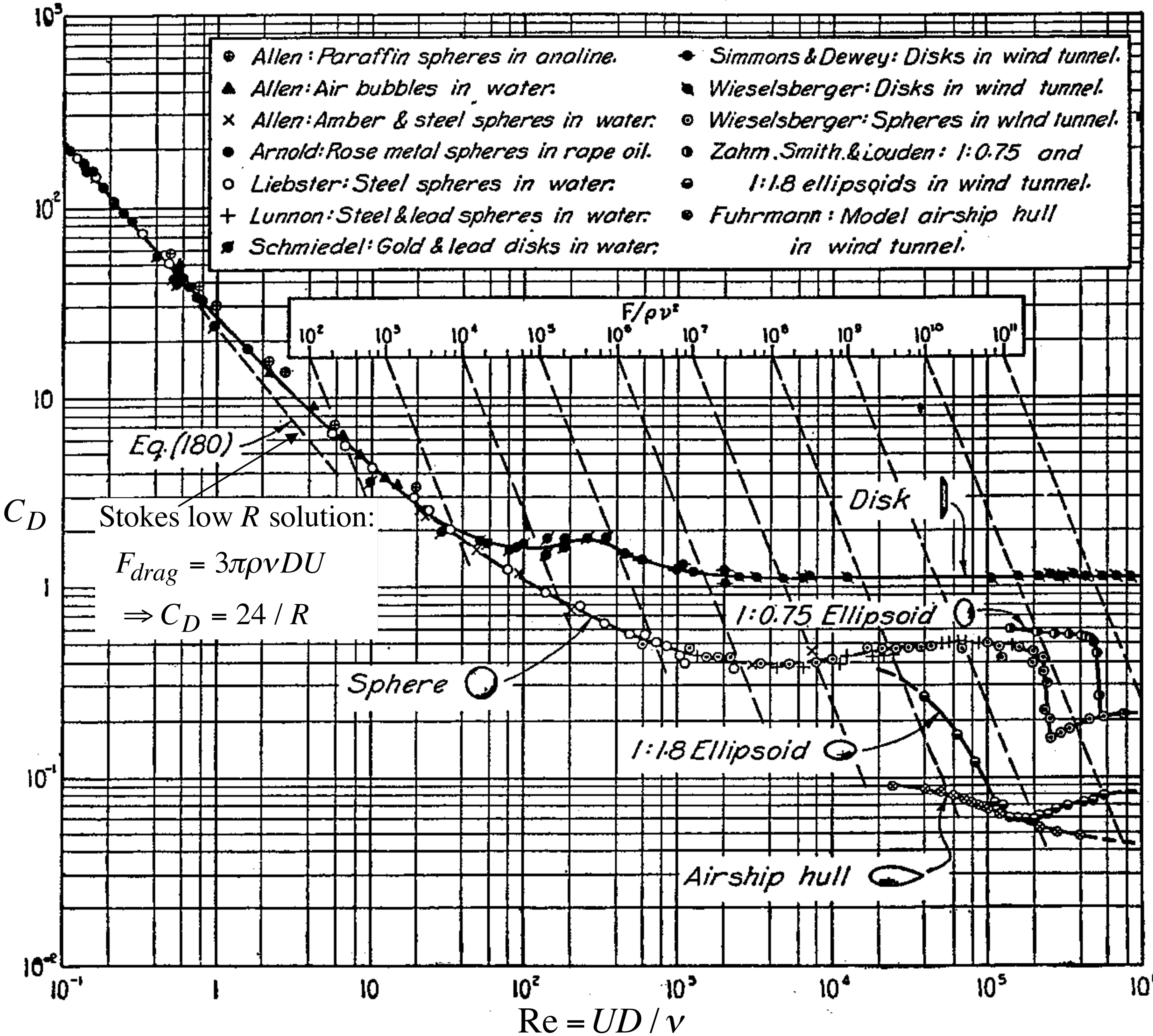


FIG. 125. Coefficients of drag as functions of the Reynolds number for bodies of revolution.

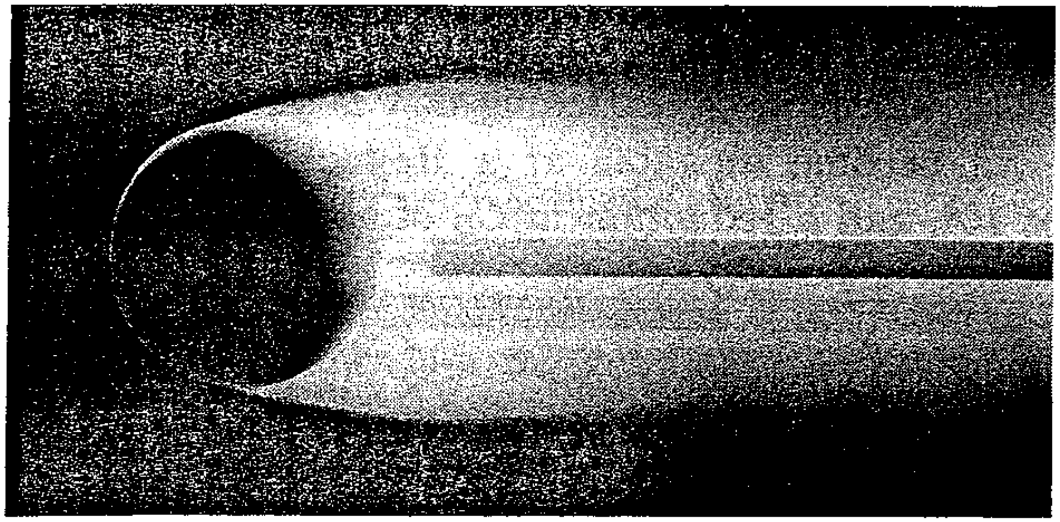
Rouse, *Elementary Mechanics of Fluids*, 1946, 1978

Dimensional analysis, drag force on a sphere of diameter D in flow at far-field velocity U:

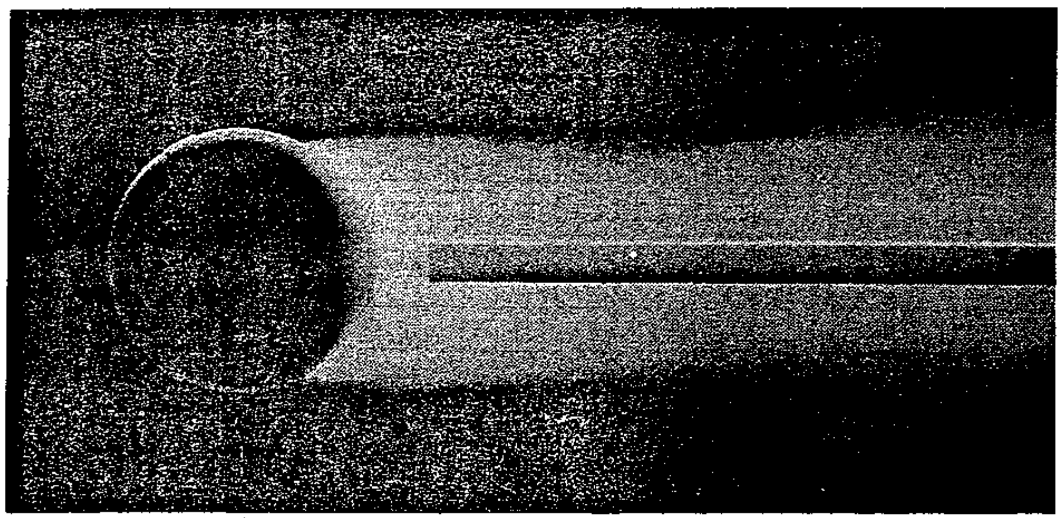
$$F_{drag} = \frac{1}{2}\rho U^2 \frac{\pi D^2}{4} C_D$$

where coef. of drag is

$$C_D = C_D(\text{Re}), \ \text{Re} = \frac{UD}{\nu}$$

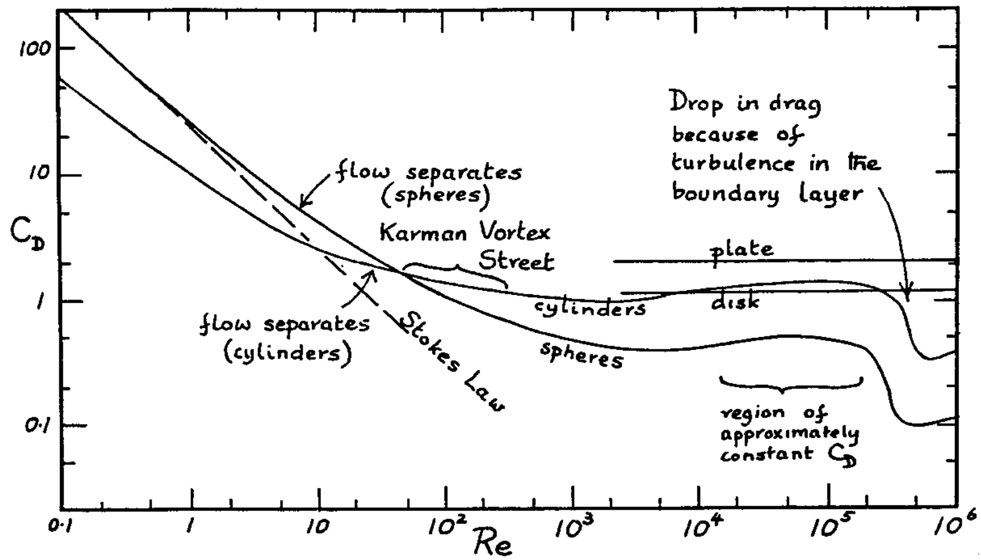


Smooth-surfaced sphere:
Promotes *laminar* flow
in *thin* boundary layer

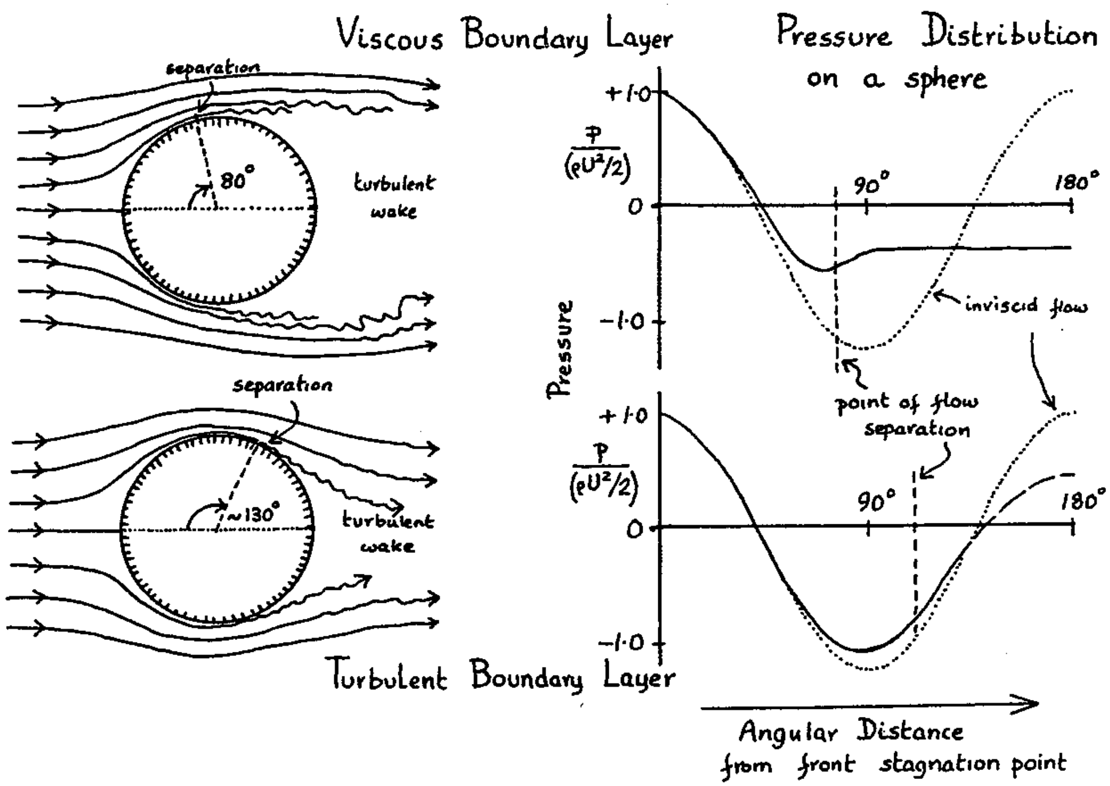


Rough-surfaced sphere:
Promotes *turbulent* flow
in *thicker* boundary layer
(*separation is delayed,*
drag is reduced)

PLATE XVI. Comparison of the wakes produced by smooth and roughened spheres in an air stream at the same Reynolds number ($\mathbf{R} \approx 10^5$), made visible by smoke injected through the tube surrounding the supporting rod. Above, the boundary layer is laminar; below, boundary-layer turbulence is induced by sand grains cemented in a narrow band around the front of the sphere.



100
10
1
0·1
C_D
0·1
1
10
10^2
10^3
10^4
10^5
10^6
Re
flow separates (spheres)
Karman Vortex Street
flow separates (cylinders)
Stokes Law
cylinders
spheres
plate
disk
Drop in drag because of turbulence in the boundary layer
region of approximately constant C_D



Viscous Boundary Layer
separation
80°
turbulent wake
separation
~130°
turbulent wake
Turbulent Boundary Layer
Pressure Distribution on a sphere
+1·0
$\frac{p}{(\rho U^2/2)}$
0
−1·0
90°
180°
inviscid flow
Pressure
point of flow separation
+1·0
$\frac{p}{(\rho U^2/2)}$
0
−1·0
90°
180°
Angular Distance from front stagnation point

Boundary layer on a thin flat plate (H. Schlicting & K. Gersten, 1955, 2000)

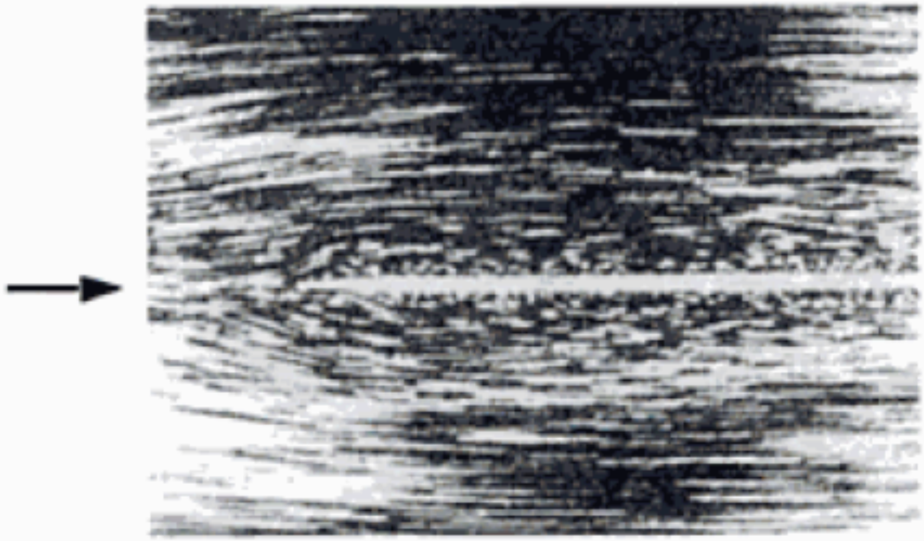


Fig. 2.1. Flow along a thin flat plate, after L. Prandtl; O. Tietjens (1931)

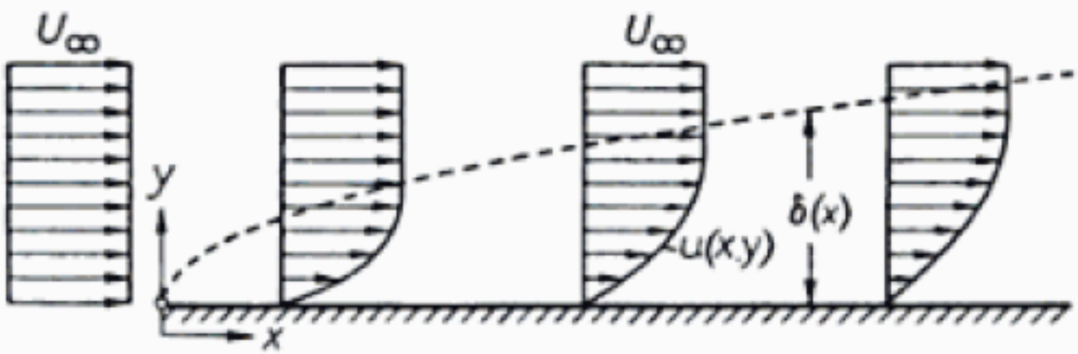
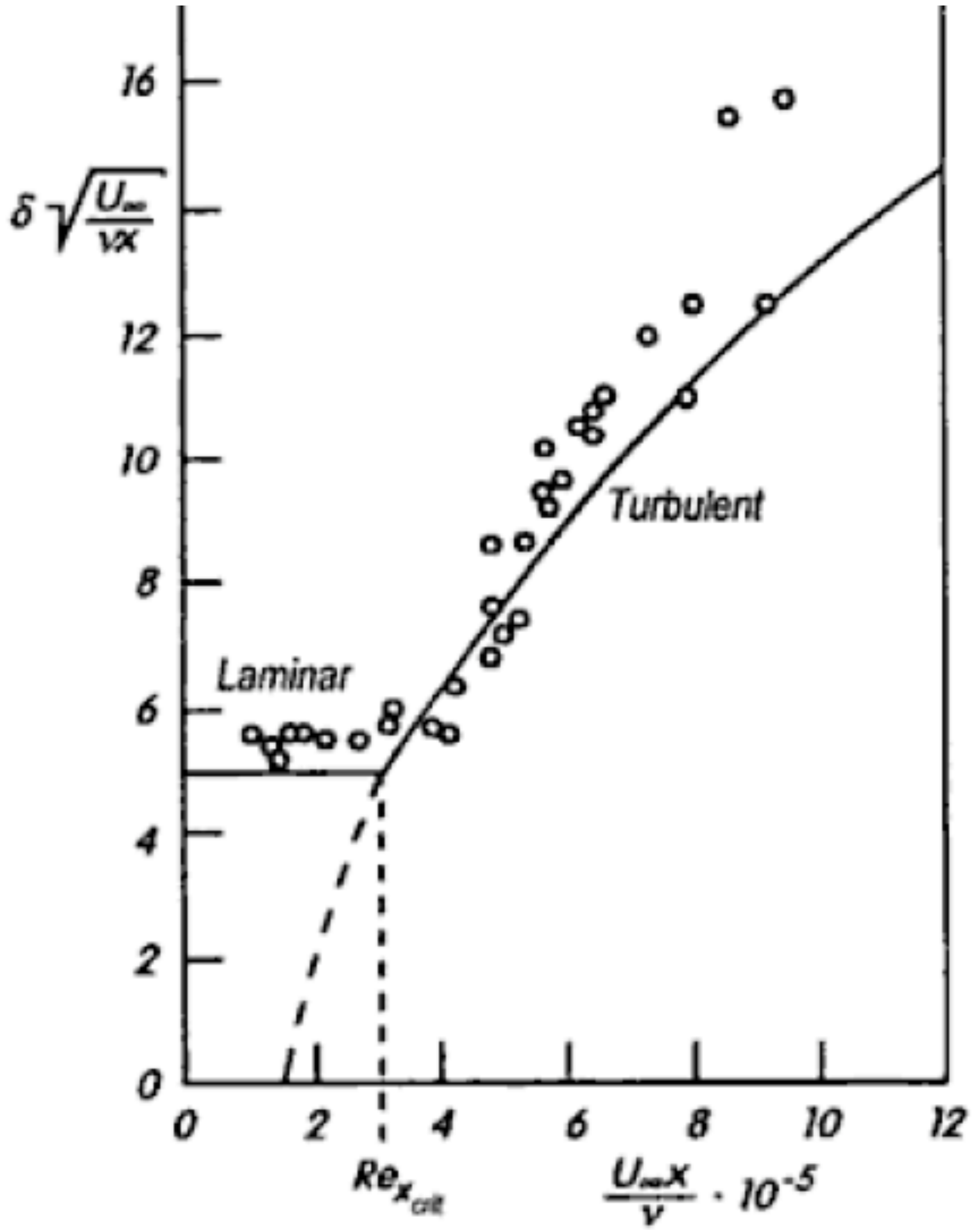


Fig. 2.2. Boundary layer at a flat plate at zero incidence (schematic)



Boundary layer (BL) thickness δ on a thin flat plate: Thin laminar BL, and transition to thicker turbulent BL.
(H. Schlicting & K. Gersten, 1955, 2000)

Fig. 2.4. Dependence of the boundary-layer thickness on the distance along a plate at zero incidence, after M. Hansen (1928)

Middleton and Southard, *Mechanics of Sediment Movement, 1984*

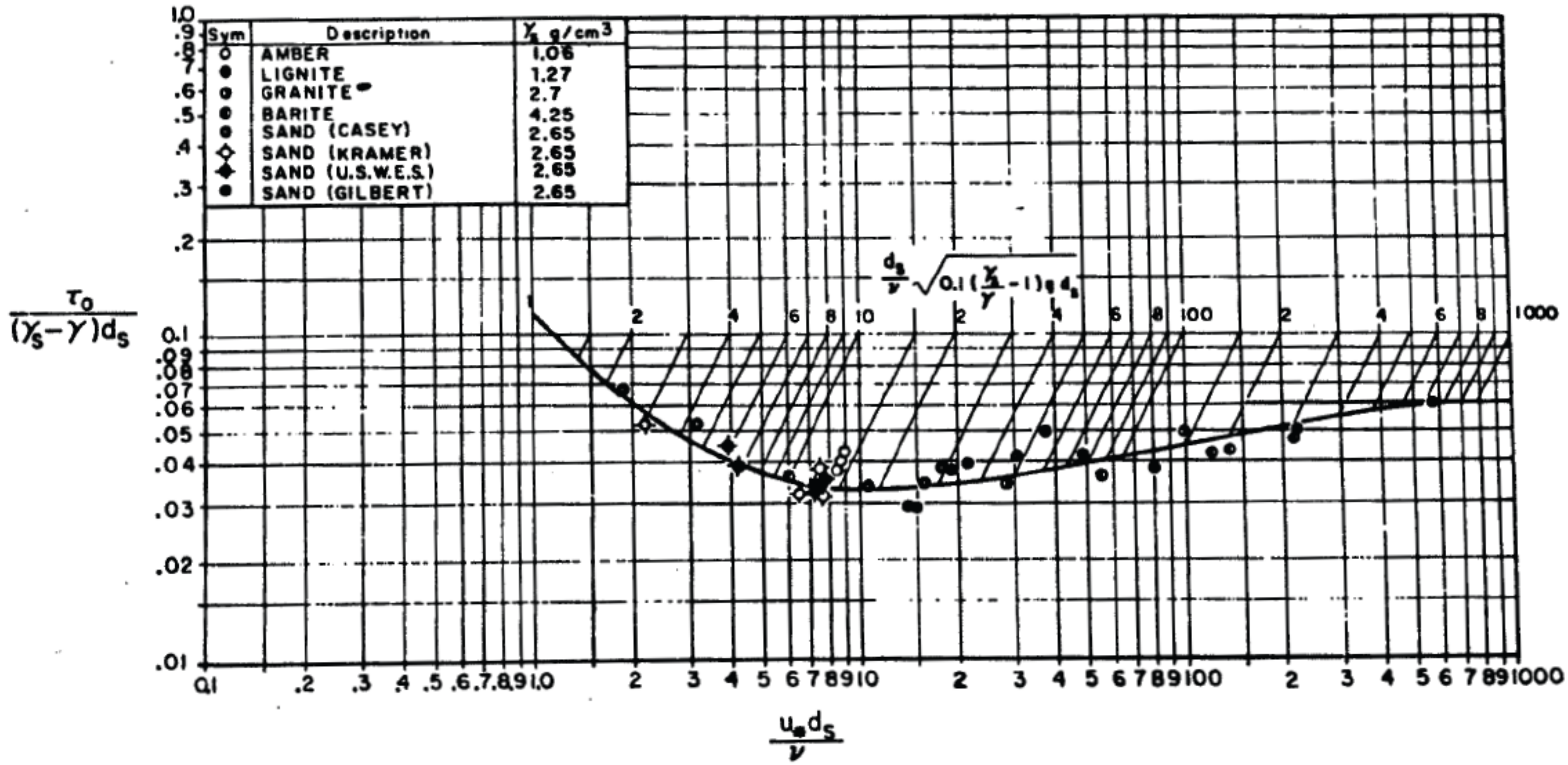
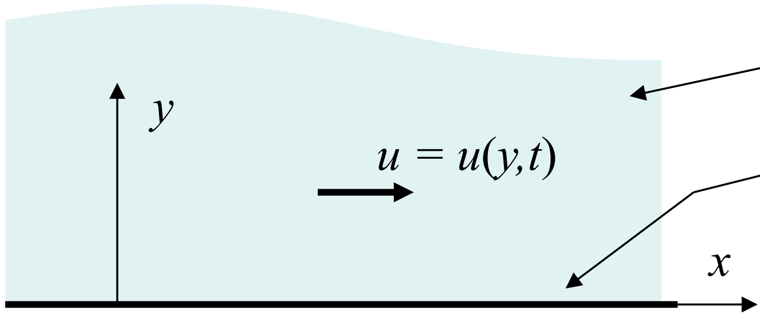


Figure 6.8 The Shields diagram, as modified by Vanoni (1964). To find the shear stress required to move a given sediment

Viscosity and the diffusion of momentum from a moving boundary



$$\frac{\partial \sigma_{yx}(y,t)}{\partial y} = \rho a_x(y,t), \quad \sigma_{yx} = \mu \frac{\partial u(y,t)}{\partial y}, \quad a_x = \frac{\partial u(y,t)}{\partial t} \quad \left(\text{here, } (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{0}\right)$$

$$\therefore \quad \nu \frac{\partial^2 u(y,t)}{\partial y^2} = \frac{\partial u(y,t)}{\partial t}$$, a *diffusion* PDE, where $\nu \equiv \frac{\mu}{\rho} =$ kinematic viscosity $[\mathrm{L}^2/\mathrm{T}]$

The problem has no characteristic length or time, so it is required from dimensional considerations (and linearity of the PDE), that $u(y,t) = u_o F(\eta)$ where $\eta = y/(2\sqrt{\nu t})$. Thus

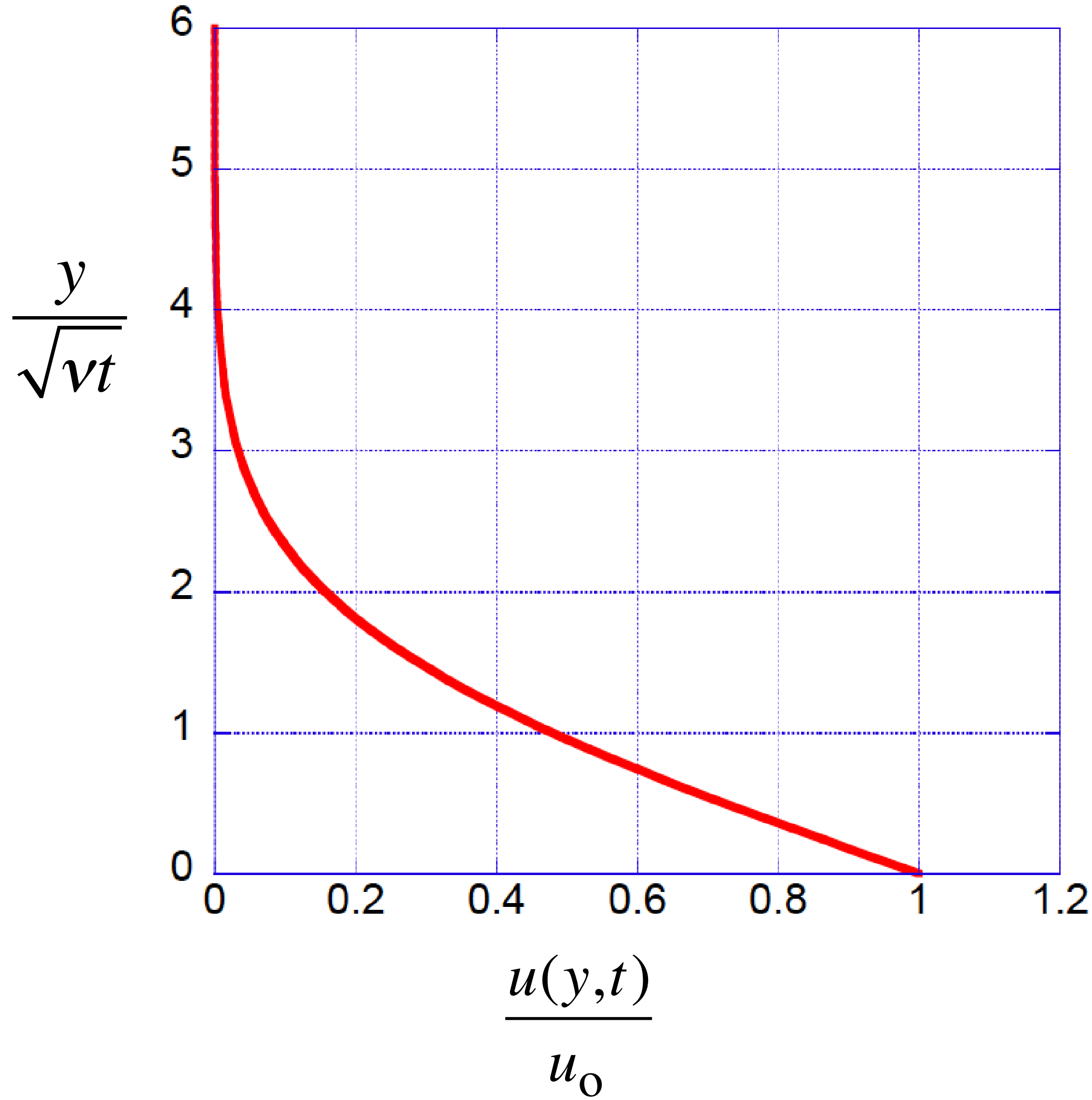
$$\nu \partial^2 u(y,t)/\partial y^2 = u_o F''(\eta)/(4t) \ \& \ \partial u(y,t)/\partial t = -u_o \eta F'(\eta)/(2t) \Rightarrow F''(\eta) + 2\eta F'(\eta) = 0$$

$$\Rightarrow \ dF'(\eta)/F'(\eta) + 2\eta d\eta = 0 \ \Rightarrow \ F'(\eta) = (2/\sqrt{\pi})\mathrm{A}e^{-\eta^2} \ \Rightarrow \ F(\eta) = \mathrm{Aerf}(\eta) + B \ (A, B \text{ const.}).$$

Condition $u \to 0$ as $y \to \infty$ and boundary condition $u = u_o$ at $y = 0 \ \Rightarrow \ B = 1$ and $A = -1$.

$$\therefore \ u(y,t) = u_o\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{y/(2\sqrt{\nu t})} e^{-\eta^2} d\eta\right) = \frac{2}{\sqrt{\pi}} u_o \int_{y/(2\sqrt{\nu t})}^{\infty} e^{-\eta^2} d\eta = u_o \mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

$$u(y,t) = \frac{2}{\sqrt{\pi}} u_o \int_{y/(2\sqrt{\nu t})}^{\infty} e^{-\eta^2} d\eta = u_o \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)$$



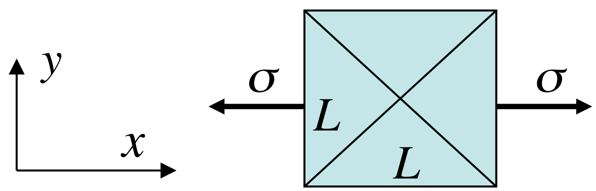
Stress at boundary:

$$\sigma_{yx}(0,t) = -\mu \frac{u_o}{\sqrt{\pi \nu t}} = -u_o \sqrt{\frac{\rho \mu}{\pi t}}$$

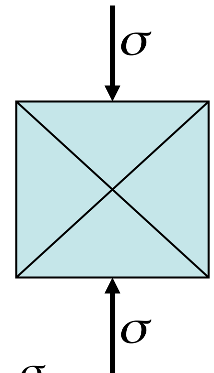
Under pure shear stress τ : $\dot{\gamma} = \tau / \mu$ where μ is the *shear* viscosity.

Under pure uniaxial tensile stress σ : $\dot{\varepsilon} = \sigma / \alpha$ where α is the *extensional* viscosity.

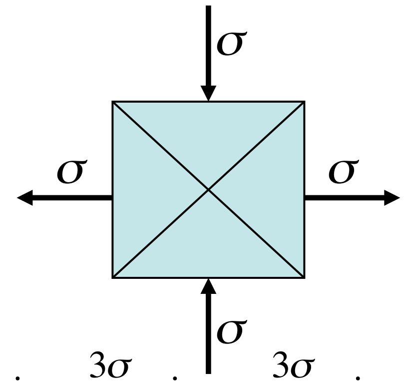
What relation between α and μ ?



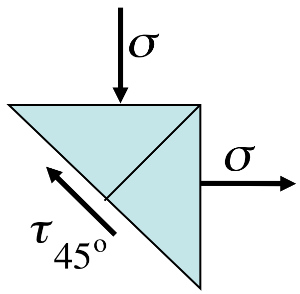
$$\dot{\varepsilon}_x = \frac{\sigma}{\alpha}, \ \dot{\varepsilon}_y = \dot{\varepsilon}_z = -\frac{\sigma}{2\alpha}$$

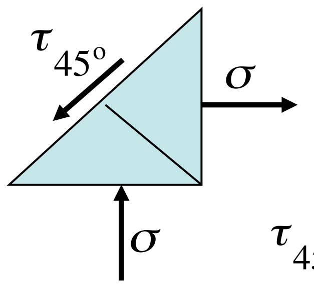


$$\dot{\varepsilon}_y = -\frac{\sigma}{\alpha}, \ \dot{\varepsilon}_x = \dot{\varepsilon}_z = +\frac{\sigma}{2\alpha}$$



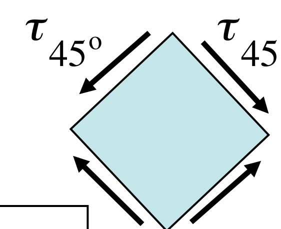
$$\dot{\varepsilon}_x = \frac{3\sigma}{2\alpha}, \ \dot{\varepsilon}_y = -\frac{3\sigma}{2\alpha}, \ \dot{\varepsilon}_z = 0$$

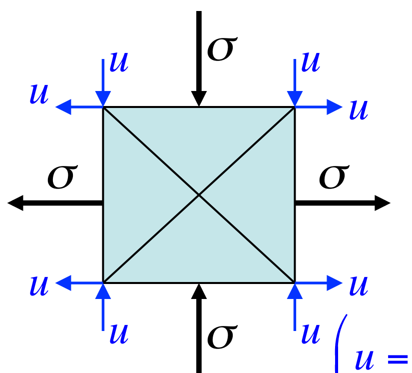


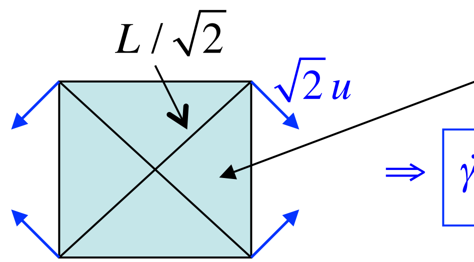


$$\sum (\text{Forces})_{45^\circ} = 0 \ \Rightarrow$$

$$\tau_{45^\circ}(\sqrt{2}L) = \frac{1}{\sqrt{2}}(\sigma L) + \frac{1}{\sqrt{2}}(\sigma L) \ \Rightarrow \ \boxed{\tau_{45^\circ} = \sigma}$$







$$\left(u = \frac{L}{2}\dot{\varepsilon}_x \ , \ u = -\frac{L}{2}\dot{\varepsilon}_y \right)$$

$$\dot{\gamma}_{45^\circ} = 2 \times \frac{\sqrt{2}\,u}{L/\sqrt{2}} = 4\frac{u}{L}$$

$$\Rightarrow \boxed{\dot{\gamma}_{45^\circ} = \dot{\varepsilon}_x - \dot{\varepsilon}_y} \ \Rightarrow \ \dot{\gamma}_{45^\circ} = \frac{3\sigma}{\alpha}$$

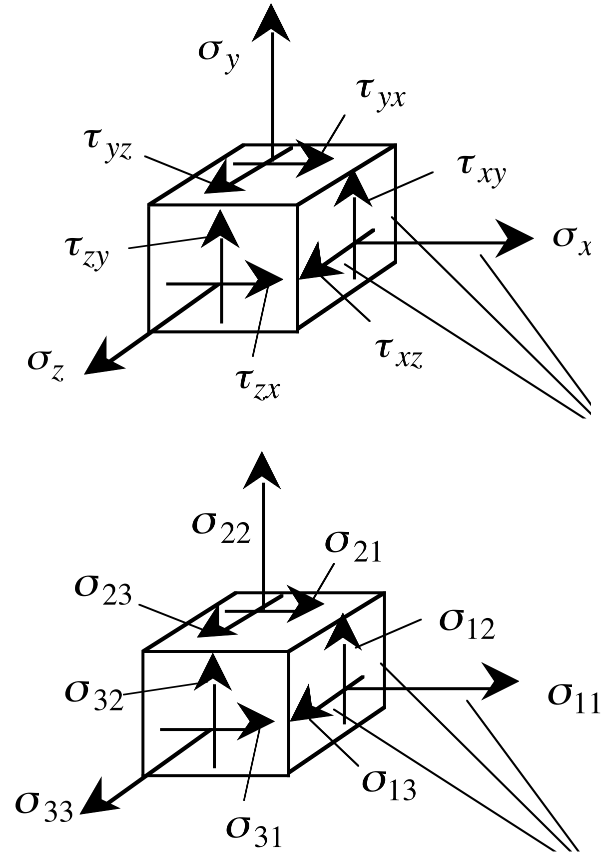
But $\dot{\gamma}_{45^\circ} = \dfrac{\tau_{45^\circ}}{\mu} = \dfrac{\sigma}{\mu} \ \Rightarrow \ \boxed{\boxed{\alpha = 3\mu}}$

Equations of motion, linear momentum, for all directions:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = \rho a_x \ ,$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = \rho a_y \ ,$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = \rho a_z$$



Equivalent to $\displaystyle \sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} + f_j = \rho a_j$

(for $j = 1,2,3$)

Equations of motion, angular momentum:

$$\tau_{xy} = \tau_{yx} \ , \ \tau_{yz} = \tau_{zy} \ , \ \tau_{zx} = \tau_{xz}$$

Equivalent to $\sigma_{ij} = \sigma_{ji}$ (for all $i, j = 1,2,3$)

The *Newtonian-viscous constitutive relation* for a simple incompressible fluid is

$$\sigma_{ij} = -p\delta_{ij} + 2\mu D_{ij} \text{ , where } D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ for } i, j = 1, 2, 3.$$

(1) δ_{ij} is called the Kronecker delta, or sometimes the unit tensor, and it takes the values

$$\delta_{ij} = 1 \text{ if } i = j \text{ , and } \delta_{ij} = 0 \text{ if } i \neq j \text{ ;}$$

(2) μ is the fluid *viscosity* ;

(3) p is *pressure*, corresponding to

$$p = -\frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \text{ [or to } p = -\frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \text{ in the elementary notation]}$$

(4) D_{ij} is called the *rate of straining* tensor; e.g., $D_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$ corresponds to $\frac{1}{2}\dot{\gamma}_{xy}$,

where $\dot{\gamma}_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$ is the (engineering) shear strain rate in the elementary notation, whereas $D_{11} = \frac{\partial u_1}{\partial x_1}$ corresponds to the extensional strain rate $\dot{\varepsilon}_x = \frac{\partial u_x}{\partial x}$.

The expression $\sigma_{ij} = -p\delta_{ij} + 2\mu D_{ij}$ then tells that, e.g., $\sigma_{12} = 2\mu D_{12}$ (equivalent to the familiar $\tau_{xy} = \mu\dot{\gamma}_{xy}$), and that $\sigma_{11} - \dfrac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \equiv \dfrac{2\sigma_{11} - \sigma_{22} - \sigma_{33}}{3} = 2\mu D_{11}$ (or $\dfrac{2\sigma_x - \sigma_y - \sigma_z}{3} = 2\mu\dot{\varepsilon}_x$). Note that the last expressions shows that for uniaxial stress, σ_x, acting on an incompressible fluid, $\sigma_x = 3\mu\dot{\varepsilon}_x$; meaning that the "extensional viscosity" of an incompressible fluid is three times the "shear viscosity".

That relation between shear and extensional viscosities follows because the fluid is *isotropic*, and is assumed to be incompressible, *and* because the stress σ_{ij} and rate of deformation D_{ij} are *second rank tensors*. The same features of σ_{ij} and D_{ij} that make them tensors are called upon, in the elementary derivation given earlier, in relating stress and strain rates relative to an original set of Cartesian axes to those (e.g., τ_{45} and γ_{45}) relative to axes rotated by 45°.

Now we combine the ***constitutive relation*** and the ***equations of motion*** the latter of which, to remind, are:

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} + f_j = \rho a_j \;,\quad a_j = \frac{\partial u_j}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial u_j}{\partial x_i} \;,\quad f_j = \rho g_j = -\rho g \frac{\partial z_{el}}{\partial x_j}$$

$$\therefore \sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} - \rho g \frac{\partial z_{el}}{\partial x_j} = \rho \left[\frac{\partial u_j}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial u_j}{\partial x_i} \right]$$

Equations of Motion for an Incompressible Viscous Fluid
(the Navier-Stokes Equations)

$$\sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} - \rho g \frac{\partial z_{el}}{\partial x_j} = \rho \left[\frac{\partial u_j}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial u_j}{\partial x_i} \right] \quad , \quad \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0$$

$$\sigma_{ij} = -p\delta_{ij} + 2\mu D_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad , \quad \sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} = -\frac{\partial p}{\partial x_j} + \mu \sum_{i=1}^{3} \frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

$$\therefore \quad -\frac{\partial \left(p + \rho g z_{el} \right)}{\partial x_j} + \mu \sum_{i=1}^{3} \frac{\partial^2 u_j}{\partial x_i \partial x_i} = \rho \left[\frac{\partial u_j}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial u_j}{\partial x_i} \right] \quad , \quad \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0$$

Vector form: $$-\nabla \left(p + \rho g z_{el} \right) + \mu \nabla^2 \boldsymbol{u} = \rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + \left(\boldsymbol{u} \cdot \nabla \right) \boldsymbol{u} \right] \quad , \quad \nabla \cdot \boldsymbol{u} = 0$$

Statistically Steady Turbulent Flow:

$$\text{Velocity field} \quad u_j = \overline{u}_j(x_1, x_2, x_3) + u'_j(x_1, x_2, x_3, t) \ ;$$

$$\text{Mean flow field:} \ \overline{u}_j(x_1, x_2, x_3) = \left\langle u_j(x_1, x_2, x_3, t) \right\rangle$$

$$\text{Fluctuation field:} \ u'_j(x_1, x_2, x_3, t); \ \left\langle u'_j(x_1, x_2, x_3, t) \right\rangle = 0$$

$$\left\langle u_i u_j \right\rangle = \overline{u}_i \overline{u}_j + \left\langle u'_i u'_j \right\rangle$$

$$-\frac{\partial p}{\partial x_j} - \rho g \frac{\partial z_{el}}{\partial x_j} + \mu \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) = \rho \left[\frac{\partial u_j}{\partial t} + \sum_{i=1}^{3} \frac{\partial \left(u_i u_j \right)}{\partial x_i} \right]$$

when time-averaged becomes

$$\sum_{i=1}^{3} \frac{\partial \overline{\sigma}_{ij}}{\partial x_i} - \rho g \frac{\partial z_{el}}{\partial x_j} = \rho \sum_{i=1}^{3} \overline{u}_i \frac{\partial \overline{u}_j}{\partial x_i} \ , \quad \text{where}$$

$$\overline{\sigma}_{ij} = -\overline{p}\delta_{ij} + \mu \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) - \rho \left\langle u'_i u'_j \right\rangle$$

(last term: ***Reynolds stress***)

vonKarman - Prandtl law of the wall

$$\frac{\bar{u}}{u*} \approx \frac{1}{0.40} \ln\left(\frac{\rho u* y}{\mu}\right) + 5.5$$

(smooth wall version)

$$\frac{\bar{u}}{u*} = \frac{\rho u* y}{\mu}$$

$$\left(\bar{u} = \frac{\tau_{wall}}{\mu} y\right)$$

$$\left(u* \equiv \sqrt{\frac{\tau_{wall}}{\rho}}\right)$$

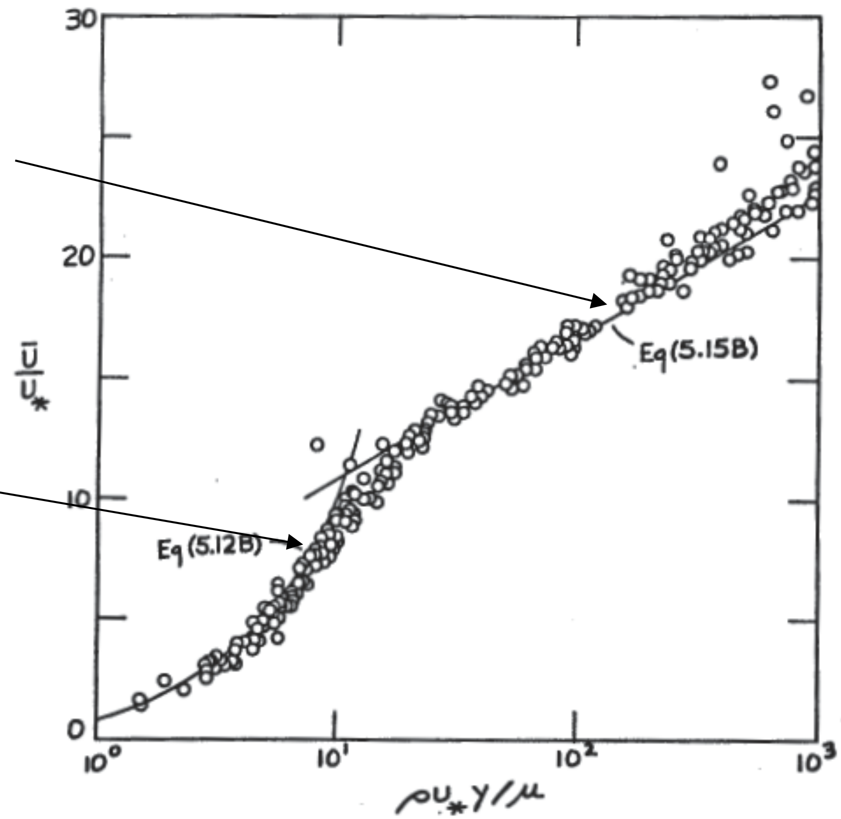


Figure 5.13 Plot of $\bar{u}/u_*$, ratio of local time-average velocity to shear velocity, against the logarithm of $\rho u_* y/\mu$, dimensionless distance from the boundary, for the inner layer over smooth boundaries. This plot represents the complete law of the wall for dynamically smooth turbulent boundary layers. Equations (5.12B) and (5.15B), for the viscosity-dominated and turbulence-dominated parts of the inner layer, are given by the solid curves. Data points are from several sources, and cover a fairly wide range of mean-flow Reynolds numbers and various outer-layer flow geometries. The intersection of Equations (5.12B) and (5.15B) at δ' is discussed in the text.

Middleton and Southard, *Mechanics of Sediment Movement, 1984*

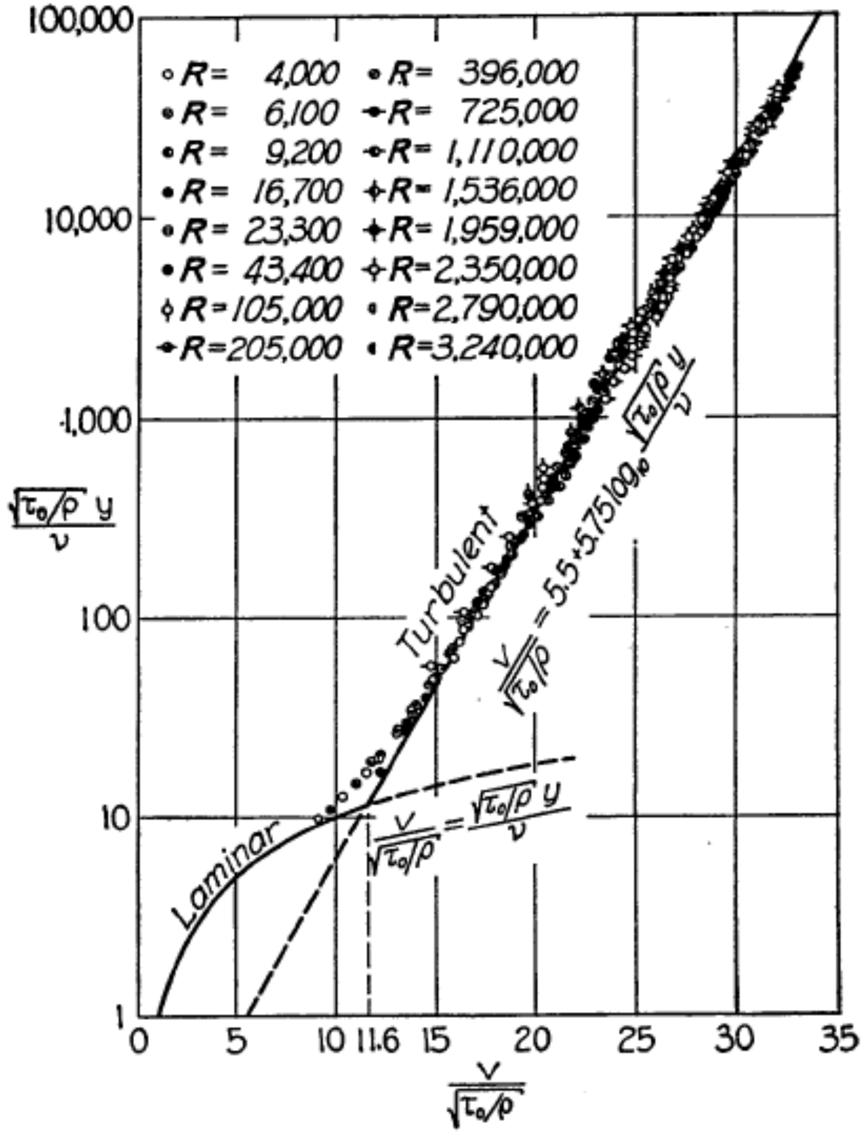


FIG. 102. Experimental verification of the Kármán-Prandtl equation for the velocity distribution near a smooth boundary.

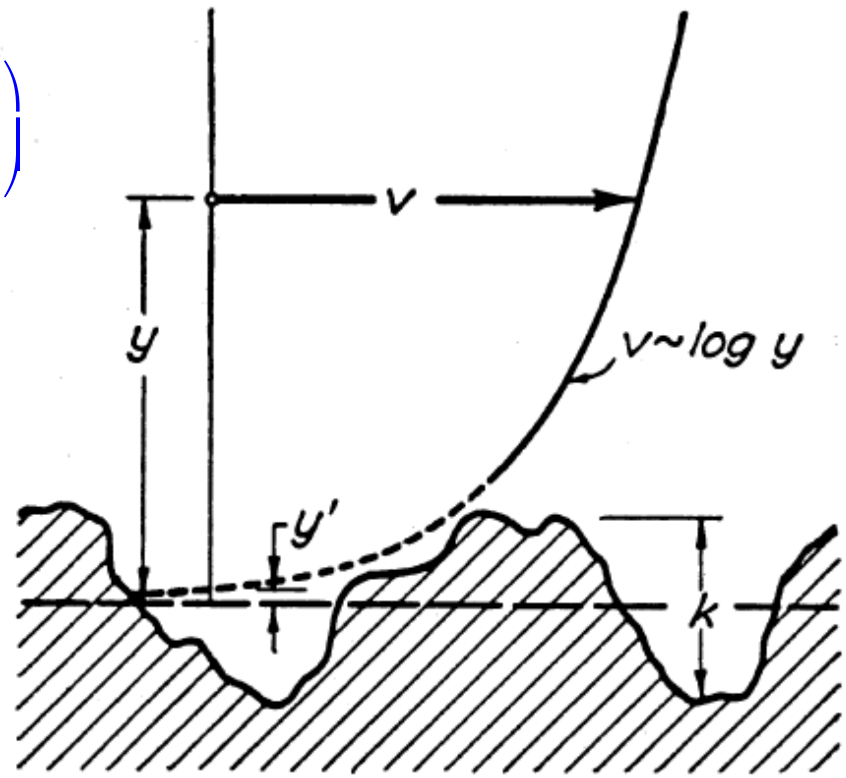
Rouse, *Elementary Mechanics of Fluids*, 1946, 1978

vonKarman - Prandtl law of the wall

$$\frac{\bar{u}}{u*} \approx \frac{1}{0.40} \ln\left(\frac{30y}{k}\right)$$

(rough wall version)

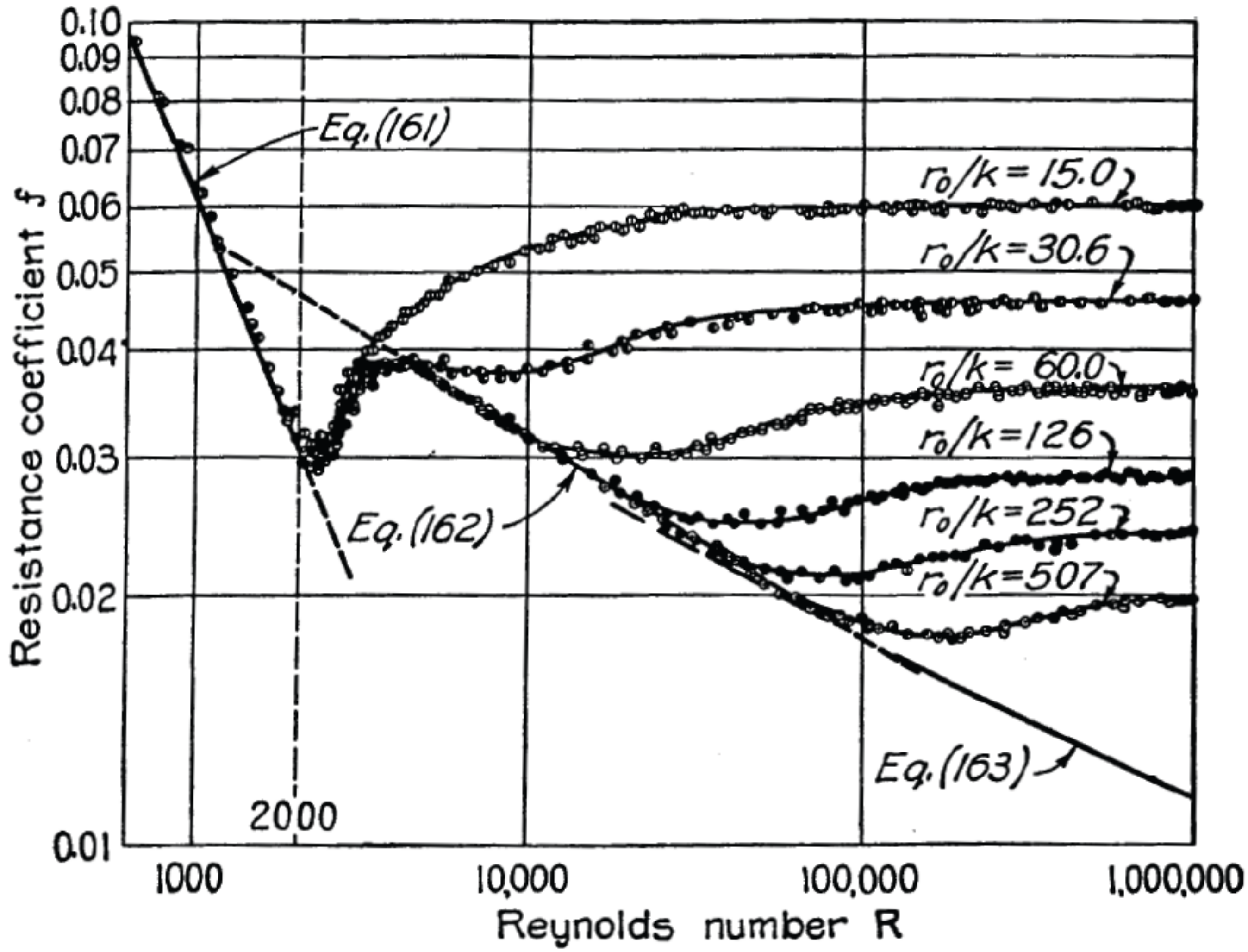
$$\left(u* \equiv \sqrt{\frac{\tau_{wall}}{\rho}} \right)$$



Rouse, *Elementary Mechanics of Fluids*, 1946, 1978

FIG. 103. Definition sketch for evaluating the relative magnitude of boundary roughness.

Rouse, *Elementary Mechanics of Fluids*, 1946, 1978



Darcy-Wiesbach friction factor f :

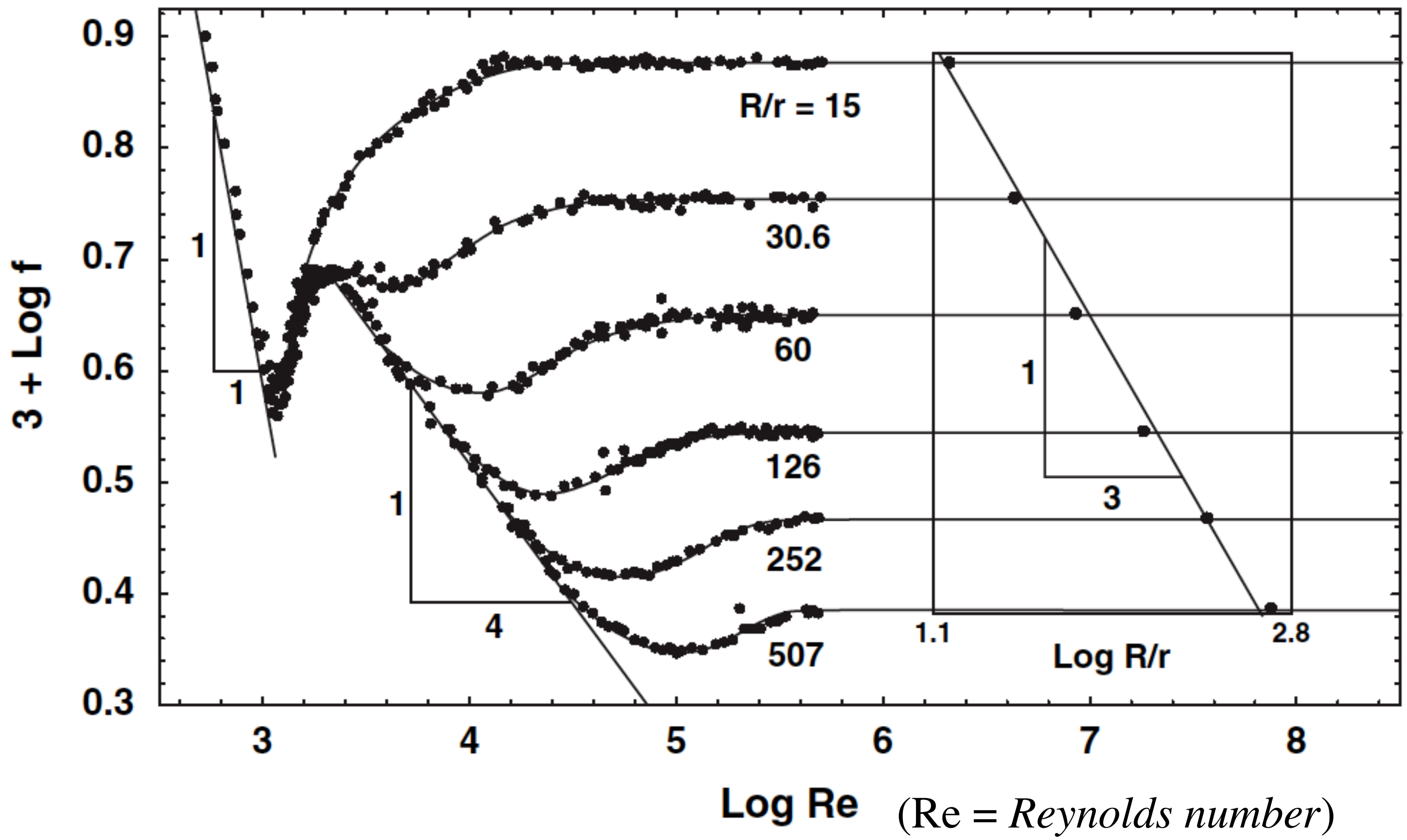
$$f = \frac{\tau_{wall}}{\rho U^2 / 8}$$

In pipe flow:

$$f = \frac{Diam}{Length} \frac{\Delta p}{\rho U^2 / 2}$$

FIG. 108. Variation of the resistance coefficient with the Reynolds number for artificially roughened pipes. (r_o = pipe radius, k = roughness)

Gioia & Chakraborty [*PRL*, 2006] replot of Johann Nikuradse [1933] pipe-flow data



(Re = *Reynolds number*)

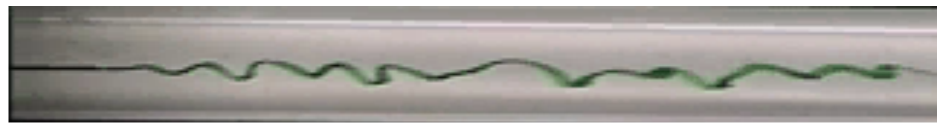
R/r
→ *h* / *k*

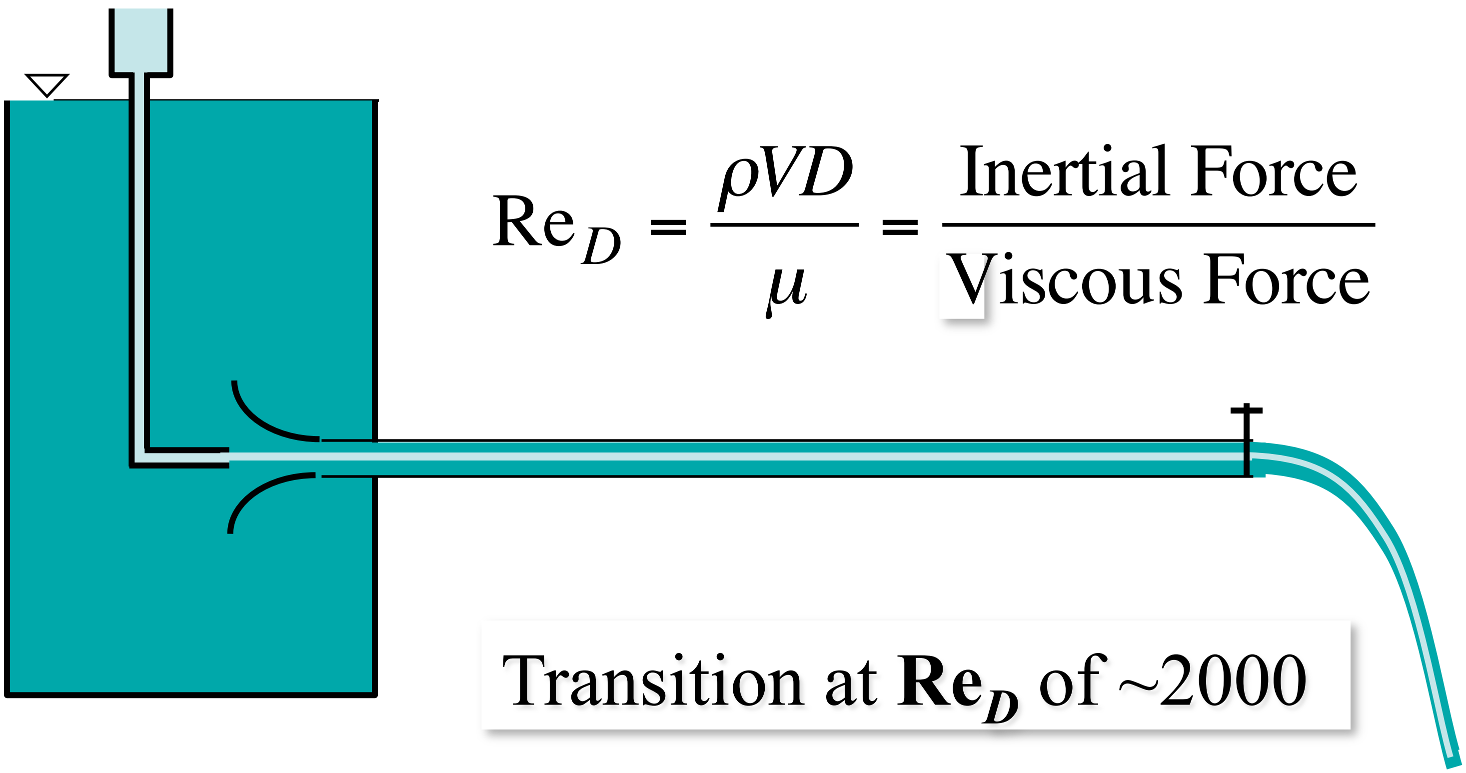
FIG. 1. Nikuradse's data. Up to a Re of about 3000 the flow is streamlined (free from turbulence) and $f \sim 1/\mathrm{Re}$. Note that for very rough pipes (small R/r) the curves do not form a belly at intermediate values of Re. Inset: verification of Strickler's empirical scaling for f at high Re, $f \sim (r/R)^{1/3}$.

Following materials (to end of this file) were excerpted from web site of supplemental materials of Prof. ***Kuang-An Chang***, Dept. of Civil Engin., Texas A&M Univ., for his spring 2008 course CVEN 311, *Fluid Dynamics*. (Various edits made by JRR.)

Laminar and Turbulent Flows

- Reynolds apparatus





$$Re_D = \frac{\rho V D}{\mu} = \frac{\text{Inertial Force}}{\text{Viscous Force}}$$

Transition at $\mathbf{Re}_D$ of ~2000

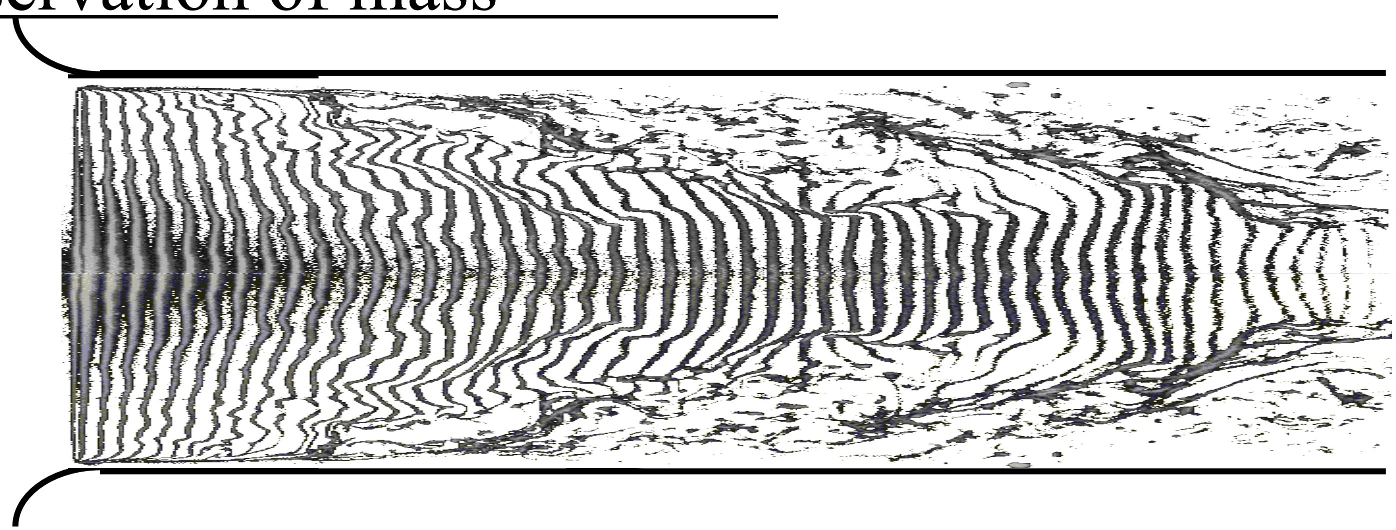
Boundary layer growth: Transition length

What does the water near the pipeline wall experience?

Drag or shear

Why does the water in the center of the pipeline speed up? _Conservation of mass_

Pipe Entrance



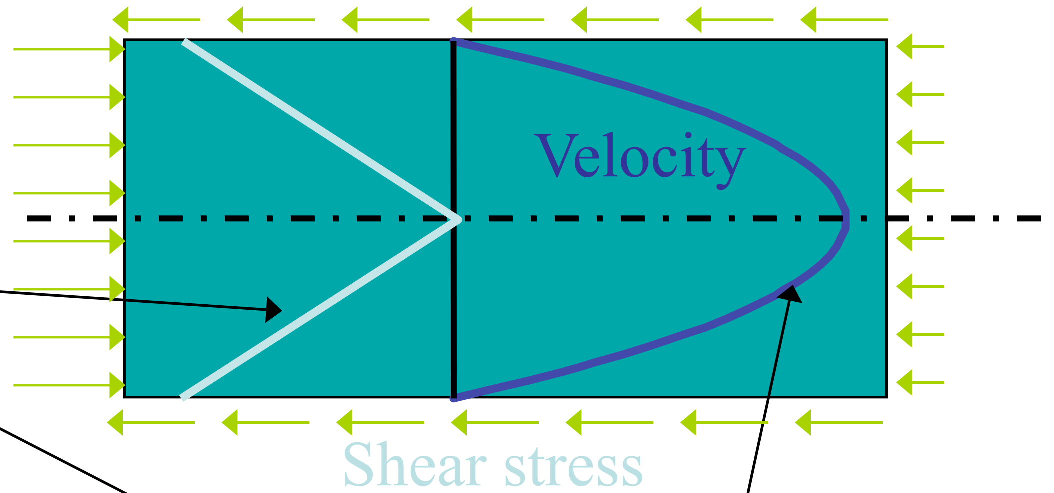
Non-Uniform Flow

Need equation for entrance length here

Flow through Circular Tubes: Diagram

$$\tau = \frac{r}{2}\frac{d}{dx}\left(p + \gamma z_{el}\right)$$

$$= -r\left(\frac{\gamma h_l}{2L}\right)$$



$$h_l \equiv \left(\frac{p}{\gamma} + z_{el}\right)_{x=0} - \left(\frac{p}{\gamma} + z_{el}\right)_{x=L}$$

τ expressions same for laminar or turbulent flow

Shear stress at the wall: $\tau_{wall} = -\frac{\gamma h_l d}{4l}$

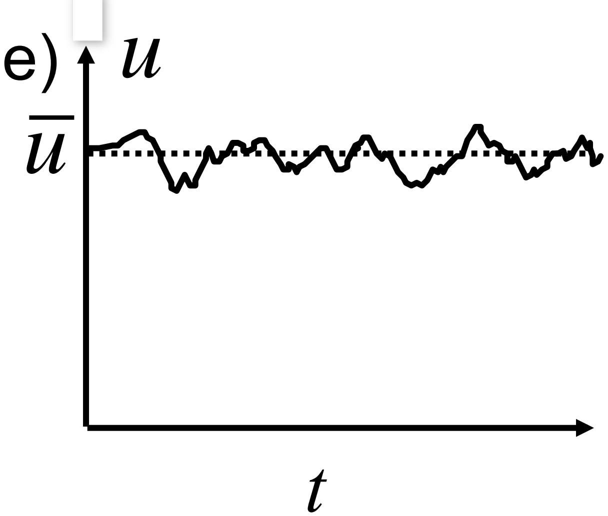
Flow velocity, ***laminar*** case: $\frac{du}{dr} = \frac{\tau}{\mu} = \frac{r}{2\mu}\frac{d}{dx}\left(p + \gamma z_{el}\right) \Rightarrow u = -\frac{a^2 - r^2}{4\mu}\frac{d}{dx}\left(p + \gamma z_{el}\right)$

Turbulence

- A characteristic of the flow.
- How can we characterize turbulence?
  - intensity of the velocity fluctuations
  - size of the fluctuations (length scale)

$$u = \overline{u} + u'$$

instantaneous velocity — mean velocity — velocity fluctuation



Turbulence: Flow Instability

- In turbulent flow (high Reynolds number) the force leading to stability (_viscosity_) is small relative to the force leading to instability (_inertia_).
- Any disturbance in the flow results in large scale motions superimposed on the mean flow.
- Some of the kinetic energy of the flow is transferred to these large scale motions (eddies).
- Large scale instabilities gradually lose kinetic energy to smaller scale motions.
- The kinetic energy of the smallest eddies is dissipated by viscous resistance and turned into heat. (=_head loss_)

Velocity Distributions

- Turbulence causes transfer of momentum from center of pipe to fluid closer to the pipe wall.
- Mixing of fluid (transfer of momentum) causes the central region of the pipe to have relatively __constant__ velocity (compared to laminar flow)
- Close to the pipe wall eddies are smaller (size proportional to distance to the boundary)

Turbulent Flow Velocity Profile

$$\tau \neq \mu \frac{du}{dy}$$

Turbulent shear is from momentum transfer

$$\tau = \eta \frac{du}{dy}$$

η = eddy viscosity

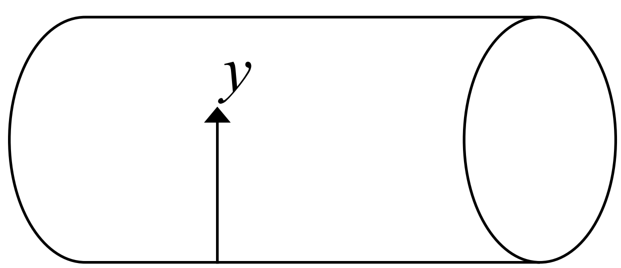
Length scale and velocity fluctuation of "large" eddies

$$\eta \propto \rho l_I u_I$$

Dimensional reasoning

$$u_I \propto l_I \frac{du}{dy}$$

$$\eta \propto \rho l_I^2 \frac{du}{dy}$$



Turbulent Flow Velocity Profile

$$\tau = \eta \frac{du}{dy}$$

$$\eta \propto \rho l_I^2 \frac{du}{dy}$$

$$l_I = \kappa y$$

Size of the eddies _increases_ as we move further from the wall.

$$\eta = \rho \kappa^2 y^2 \frac{du}{dy}$$

$\kappa = 0.4$ (from fitting to experiments)

$$\tau = \rho \kappa^2 y^2 \left(\frac{du}{dy} \right)^2$$

$$\sqrt{\frac{\tau}{\rho}} = \kappa y \left(\frac{du}{dy} \right)$$

Log Law for Turbulent, Established Flow, Velocity Profiles near Wall (Law of the Wall)

$$\sqrt{\frac{\tau}{\rho}} = \kappa y \left(\frac{du}{dy} \right)$$

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{y u_*}{\nu} + 5.5$$

Integration and empirical results ($\kappa = 0.4$)

$$u_* = \sqrt{\frac{\tau_{wall}}{\rho}}$$

Shear velocity

$$\nu = \frac{\mu}{\rho}$$

$$u_* \approx u_I$$